Probability and Likelihood

CCBB Introduction to Biostats

September 11, 2015

Recipe for a probability model

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2. Assign probabilities to outcomes (must add to 1).
3. Define dependencies among random variables.

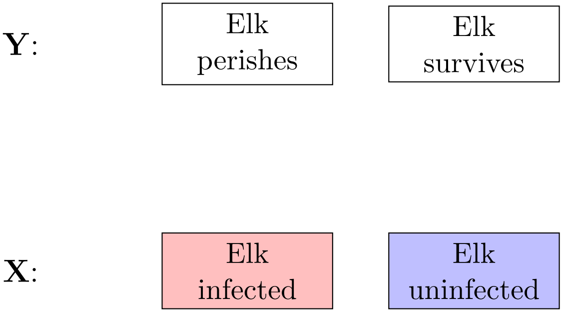
Example probability model: sick elk

Two outcomes each for random variables **X** and **Y**.

Y:	Elk perishes	Elk survives
X:	Elk infected	Elk uninfected

Example probability model: sick elk

Incidence of bovine tuberculosis in the population is 7%.

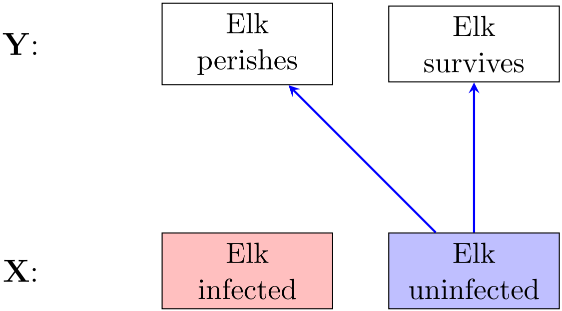


Marginal probability of variable **X** (infection status):

$$\Pr(\text{X} = \text{Infected}) = 0.07$$
$$\Pr(\text{X} = \text{Uninfected}) = 0.93$$

Example probability model: sick elk

Uninfected individuals are more likely to live than to die.



Conditional prob. of variable **Y** (survival) given **X = uninfected**:

$$\Pr(Y = \text{Survives}|X = \text{Uninfected}) = 0.77$$
$$\Pr(Y = \text{Perishes}|X = \text{Uninfected}) = 0.23$$

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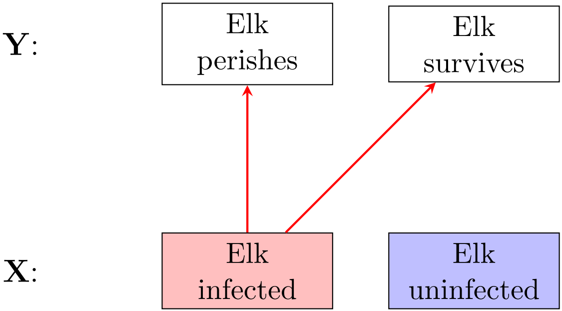
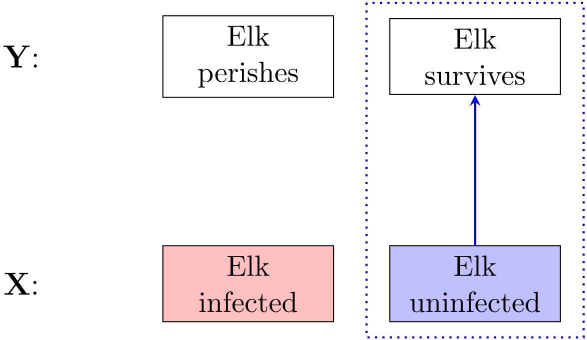


Conditional prob. of variable **Y** (survival) given **X = infected**:

$$\Pr(Y = \text{Survives} | X = \text{Infected}) = 0.17$$
$$\Pr(Y = \text{Perishes} | X = \text{Infected}) = 0.83$$

Example probability model: sick elk



Joint prob. of **Y = survival** AND **X = uninfected**:

$$\Pr(\text{Y} = \text{Survives},\ \text{X} = \text{Uninfected}) =$$
$$\Pr(\text{X} = \text{Uninfected}) \times \Pr(\text{Y} = \text{Survives} | \text{X} = \text{Uninfected}) =$$
$$0.93 \times 0.77 = 0.716$$

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What about the marginal probability of random variable $\mathbf{Y}$?

Two possible ways for outcome $\mathbf{Y} = \textbf{survive}$.

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What about the marginal probability of random variable **Y**?

Two possible ways for outcome **Y = survive**.

1. **X = uninfected** and **Y = survive**.

The marginal probability equals:

$$\Pr(\mathrm{Y} = \mathrm{Survive}) =$$

$$\Pr(\mathrm{Y} = \mathrm{Survive},\ \mathrm{X} = \mathrm{Uninfected})$$

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Two possible ways for outcome $\mathbf{Y} = \mathbf{survive}$.

1. $\mathbf{X} = \mathbf{uninfected}$ and $\mathbf{Y} = \mathbf{survive}$.
2. $\mathbf{X} = \mathbf{infected}$ and $\mathbf{Y} = \mathbf{survive}$.

The marginal probability equals:

$$\Pr(\mathrm{Y} = \mathrm{Survive}) =$$
$$\Pr(\mathrm{Y} = \mathrm{Survive},\ \mathrm{X} = \mathrm{Uninfected}) + \Pr(\mathrm{Y} = \mathrm{Survive},\ \mathrm{X} = \mathrm{Infected})$$

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What about the marginal probability of random variable **Y**?

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2. **X = infected** and **Y = survive**.

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$$\Pr(\mathrm{Y} = \mathrm{Survive},\ \mathrm{X} = \mathrm{Uninfected}) + \Pr(\mathrm{Y} = \mathrm{Survive},\ \mathrm{X} = \mathrm{Infected})$$
$$= (0.93 \times 0.77) + (0.07 \times 0.17) = 0.728$$

The sum of values of the joint distribution where **Y = survive**.

Marginal, joint probability as a table

	Infected	**Uninfected**	
Survives	Pr(Infected)× Pr(Survives|Infected)	Pr(Uninfected)× Pr(Survives|Uninfected)	Pr(Survives)
Perishes	Pr(Infected)× Pr(Perishes|Infected)	Pr(Uninfected)× Pr(Perishes|Uninfected)	Pr(Perishes)
	Pr(Infected)	Pr(Uninfected)	1

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$$\Pr(X,\ Y) = \Pr(X) \times \Pr(Y|X) = \Pr(Y) \times \Pr(X|Y)$$

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Marginal probability is the probability of a random variable taking a value, *regardless* of the values of other random variables.

$$\Pr(\mathrm{Y}) = \sum_{x \in \chi} \Pr(\mathrm{Y},\ \mathrm{X} = \mathrm{x}) = \sum_{x \in \chi} \Pr(\mathrm{X} = \mathrm{x}) \times \Pr(\mathrm{Y}|\mathrm{X} = \mathrm{x})$$

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Joint probability is the probability of a *combination* of values of several random variables.

$$\Pr(\mathrm{X},\ \mathrm{Y}) = \Pr(\mathrm{X}) \times \Pr(\mathrm{Y}|\mathrm{X}) = \Pr(\mathrm{Y}) \times \Pr(\mathrm{X}|\mathrm{Y})$$

Generalizing to other types of RVs

The set of possible outcomes for a random variable can be:

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These functions are called probability distribution functions:

- aka PDFs
- with finite number of parameters
- parameters often have an intuitive interpretation
- these functions **must** sum (integrate) to 1.

Shiny time

1. Examples of probability distribution functions.
2. How does conditional/marginal/joint probability translate to continuous RVs?

Code on GitHub:

`https://github.com/sjfox/CCBB_Intro_Biostats/tree/master/week_2`

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Problem:

Have data. Need explanation.

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Likelihood measures how probable the data are at a parameter value.

Shiny time

1. Examples of calculating the likelihood.

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Measures the information the data bring about the parameters, and is used to calculate the **standard error** of the MLE.

Takeaway:

- Types of probability and how they relate.
- What a probability distribution function is.
- What likelihood is, and how to calculate it.
- What the likelihood tells us about the possible parameter values that generated our data.
- How uncertainty in the maximum likelihood estimate is measured (with the Fisher information/Hessian)