LINEAR PROGRAMMING
ORI 391Q.5 (18820)

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PREREQUISITES: An *Introduction to Operations Research* course such as ME 366M (OR Methods), ME 366L (OR Models) or equivalent; an understanding of linear algebra; a working knowledge of at least one computer programming language (e.g., C++, VBA, Java).

TEXTS Required: Dimitris Bertsimas and John N. Tsitsiklis, *Introduction to Linear Optimization*, Athena Scientific, Belmont, MA, 1997. http://www.athenasc.com/linoptbook.html

Recommended: Chapter 2, *Linear Programming,* from *Practical Bilevel Programming*, by J. F. Bard, Kluwer Academic Press, Boston, 1999

Excel Add-Ins: http://www.me.utexas.edu/~jensen/ORMM/ Follow instructions for loading and using.

OBJECTIVES: To develop a thorough and complete understanding of linear programming in order to be able to undertake more advanced work in optimization. This will be achieved by a detailed presentation of theory, a discussion of applications, and the development of related software.

GENERAL COURSE POLICIES

HOMEWORK: Homework and computer assignments can be done in teams of 3 or 4 students. Dates will be posted to indicate when these assignments are due.

SOLUTIONS: All solutions to the homework assignments and exams will be posted on the Canvas website.

EXAMS: All exams are open book and open notes. All students must take the exams when scheduled, including the final — no exceptions (dates on exams 1 and 2 are subject to minor changes). There will be **NO** make-up exams and there will be **NO** incompletes in the course; every student will get a grade. If you would like to review your final exam, please do so no later than one week after the next semester begins (fall, spring, or summer). After that time, all exams and other unreturned class material will be discarded.

GRADING:	Homework	8%
	Programs	7%
	First Exam	20%
	Second Exam	25%
	Final Exam	40%

Arithmetic errors on exams and homework are usually penalized a few points. Conceptual errors or errors in logic are penalized more severely. On exams, always show your computations and logic that led to your solution, label the solution, and cross out any work that you do not want graded. If your exam contains erroneous information or computations, even if you have the correct solution, you will still be penalized.

Necessary (but not sufficient) conditions for receiving a grade of **A** in the course are either an **A** on the first two examines or an **A** on the final and at least a high **B** on the first two exams. An **A** on the two midterms and a **C** on the final will likely result in a **B** in the course. Analogous requirements exist for receiving a grade of **B** and so on. Grades on the homeworks and programs are used to evaluate borderline students. All students **must** take the **final** on the scheduled date.

RE-GRADING: If you feel that you weren't graded fairly on an exam question, **first** look at the solution, **then** write a note to me explaining how your answer compares to the posted solution and why you think that you deserve more credit. Hand in your note and exam to me no later than 2 weeks after the exam is returned.

EXTRA CREDIT: There will be **NO** extra assignments for those wishing to try to improve their overall grade.

DISHONESTY: University policies for academic dishonesty will be strictly followed. Students found cheating on any exam will receive a grade of "F" in the course. Homework and other assignments turned in that do not represent the student's original work will receive a grade of zero.

DISABILITIES: The University of Texas provides upon request academic adjustments for students with documented disabilities. For more information, contact the Division of Diversity and Community Engagement, Services for Students with Disabilities, 512-471-6259, http://diversity.utexas.edu/disability/.

LP Homework Assignments

CHAPTER	TOPIC	PROBLEMS	HW #	Due†
1.2 1.5	Problem Formulation Review of Linear Algebra	1.9, 1.11, 1.15, 1.16, 1.17	1	9/12
1.3	Piecewise Linear Problems	1.3[a] 1.4, 1.5, 1.8	2	9/19
2, 3.6	Geometry of the	2.2, 2.6(a)[b], 2.10[c], 2.17, 2.22[d]	3	9/28
4.8-4.9	Simplex Method	4.41, 4.42, 4.44[e]	4	10/5
1.1, 3.1-3.3, 3.5	The Simplex Method	3.2, 3.4, 3.12, 3.17, 3.20[f]	5	10/12
3.3	Revised Simplex Method	3.6, 3.17[h], 3.21[i], 3.23[j]	6	10/19
4	Duality	4.1, 4.4, 4.7, 4.9, (4.19a,b[g] optional)	7	10/26
3.3, 3.7	Efficient Procedures for Computing Inverse	Handout		
4.5	Dual Simplex Algorithm	Handout[k]	8	11/9
5.1-5.4	Sensitivity Analysis	5.1, 5.2[*l*], 5.6[m], 5.8	9	11/21
5.5	Parametric Programming	5.12[n], 5.14[o]	10	
Notes	Bounded Variables	3.25[p] (optional)	11	
8.1, 9.4-9.6	Interior Point Methods	9.11, 9.12 (optional), 9.15[q]	12	11/30
7.2, 7.5, 7.9, 7.10	Network Flow Programming			
6.4	Decomposition Techniques			

†Due dates may change depending on the material covered each week.

[a] Note that $f(x) = \max\{1-x,\ 0,\ 2x-4\}$.

[b] Make use of the fact that Λ is in standard form and that a BFS must have n active constraints.

[c] For part (a), recall that a BFS is a point while a basis is a set of columns.

[d] For part (a), let $S \equiv P \times Q = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in P, \mathbf{y} \in Q\}$, let $T \equiv P + Q$ and make use of Corollary 2.5. That is, let (x,y) be x in Corollary 2.5, let z be y. (What does the A matrix in the corollary have to be, and what are the equivalences of S and T?) For part (b), make use of Definition 2.7, which says that if $\mathbf{x}$ is a vertex of some polyhedron $P \subset \Re^n$, then there exists a $\mathbf{c} \in \Re^n$ for which $\mathbf{x}$ is a unique minimizer of $\mathbf{cx}$ for all $\mathbf{x} \in P$. Consider $\mathbf{x}$ for $\mathbf{y}$ fixed and vice versa.

[e] For part (b) of Exercise 4.44, you cannot use solve the extreme homogeneous problem to find extreme rays. Put homogeneous problem in the form $\mathbf{Ad} \geq \mathbf{0}$, and solve for $\mathbf{d}$. There are two solutions. See Subsection "Rays and Recession Cones" on page 175.

[f] Part (b) of Exercise 3.20 should be "second" row of tableau; you have to consider different positive and negative values for α and β.

[g] Consider the primal problem: Minimize $\{-x_j : \mathbf{x} \in P\}$

[h] Solve by hand using the revised simplex method. Check your answer with the Linear/Integer Excel Add-in that can be downloaded from: http://www.me.utexas.edu/~jensen/ORMM/ Hand in the computer solution as well. All the Jensen add-ins should be on the computers in the undergraduate laboratory in ETC 2.126. If not, you can download them from the above URL and use them for the session. They are in a file called jensen.lib.zip and have to be removed before they can be used. Once you do this, open Excel and load the add-in called "add_ormm.xla". This should put a tab on the Excel menu called "Add-ins". Click the tab and you should see OR_MM on the left. Click OR_MM and then click "Add ORMM". A dialog box should appear. Check the Math Programming and LP/IP Solver boxes and then click "OK". Once this is done, click the OR_MM button again and you should see a line that says "_Linear/Integer…" When you click this line you will be presented with a dialog box that let's you define a linear program.

[i] Solve by hand using the revised simplex method for part (a). Also solve with Jensen's Excel add-in. For part (b), use the sensitivity results from the Excel solution to try to find the upper bound on p. Given the optimal solution, the sensitivity report tells you how large (upper bound) and how small (lower bound) each objective function and right-hand-side coefficient can get without affecting the optimality of the current basis. (Although this is all you are expected to do, the calculations will not be correct. Why?)

[j] For part (a) in Exercise 3.23, let $\mathbf{x}^*$ be the current solution such that the first m variables are basic. Now view the problem from the current point $\mathbf{x}^*$ and let the m basic variables be slacks. Offer a logical explanation why x_n has to be positive, and hence basic, in any optimal solution.

[k] For Dual Simplex problem #4, let $\boldsymbol{\pi}$, $\boldsymbol{\beta}$, $\bar{\mathbf{b}}$, $\bar{z}$ be respectively the dual basic solution, the basis inverse matrix, the current right-hand-side vector, and the objective function value. Let $\boldsymbol{\beta}_r$ be the rth row of the current basis inverse $\boldsymbol{\beta} \equiv \mathbf{B}^{-1}$. In the dual simplex algorithm the current components of row r for the nonbasic variables are given by $\bar{\mathbf{A}}_r = \boldsymbol{\beta}_r\mathbf{A}$. From the problem statement we know that $\bar{\mathbf{A}}_r \geq \mathbf{0}$ so $-\boldsymbol{\beta}_r\mathbf{A} \leq \mathbf{0}$. Now consider $\boldsymbol{\pi} - \lambda\boldsymbol{\beta}_r$ for scalar $\lambda > 0$. Write the dual constraints in vector form. What can you say about the dual feasibility of $\boldsymbol{\pi} - \lambda\boldsymbol{\beta}_r$? What is the objective function value at this point? Unbounded?

Note that there are other was to solve this problem by considering the full tableau of the primal problem being solved and writing out its dual.

[l] For Exercise 5.2(a) there are several possible answers. You try to make a reasoned argument that the determinant of $\mathbf{B} + \delta\,\mathbf{E}$ is nonzero. What does this imply? Alternatively, you can show that if the first column of $\mathbf{B}$ cannot be written as a linear combination of the other columns, then never can the first column of $\mathbf{B} + \delta\,\mathbf{E}$.

[m] Part (d) of Exercise 5.6: consider the offer by Company D to be in addition to the lamps that company A already has planned to produce.

Part (e). Should be "What is the *minimum* decrease…"

[n] For part (b) of Exercise 5.12, try to make up a 2-dimensional example, say, where $\mathbf{x}^1 = (1,0)$, $\mathbf{x}^2 = (0,0)$, $\mathbf{x}^3 = (0,1)$. You will need to introduce constraints such that these vectors are extreme points, and that for $\theta = \theta^*$, they are all optimal. (For what objective function are all feasible points optimal?) To complete the example you need to specify $\mathbf{c}$ and $\mathbf{d}$ in the objective function given by $\mathbf{c} + \theta\mathbf{d}$.

[o] For part (a) of Exercise 5.14, let $\mathbf{x}^1$ be solution when $\theta = 10$ and $\mathbf{x}^2$ the solution with $\theta = -10$. For part (d), try to make up an example with 2 variables such that $f(\theta)$ is neither convex nor concave.

[p] Replace the last sentence of part (a) with "Also, show that it is nondegenerate if and only if $x_i \neq 0$ and $x_i \neq u_i$ for every basic variable x_i."

[q] Part (b) should read: $\mathbf{d}_s^k = \mathbf{X}_k^{-1}(\mathbf{v}^k(\mu^k) - \mathbf{S}_k\mathbf{d}_x^k)$. Also for part (b), first verify that the values of $\mathbf{d}^k$ are correct by substitution into the expressions in part (a). Then try to derive the solution in part (b). Hint: one way to do this is solve for $\mathbf{d}_x^k$, then $\mathbf{d}_s^k$ and then $\mathbf{d}_p^k$ in this order. Take advantage of the fact that $\mathbf{A}\,\mathbf{d}_x^k = \mathbf{0}$.

General References

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