Definition and Properties of Absolute Value

The absolute value of a number represents its distance from 0 on the number line. Below, we show \(-7\) and 7 on a number line. By the function definition above, \(|-7| = |7| = 7\), and in the picture we see that both \(-7\) and 7 are 7 units away from 0.

We can also express the absolute value as a function, we define the absolute value of \(x\) as follows:

\[
|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}
\]

The absolute value of a number satisfies certain properties, which we can verify with the function definition or which we can intuit by viewing the absolute value of a number as its distance from 0.

1. The distance from 0 to 0 on a number line is 0.
   
   Alternatively, \(|0| = 0\) since plugging 0 into the function above yields 0.

2. The distance from any nonzero number to 0 on the number line is positive since nonzero distances are always positive (a negative distance doesn’t make sense).
   
   Furthermore, if \(a \neq 0\), then \(|a| > 0\). By the function definition above, we see that \(|x|\) is positive whenever \(x > 0\) since \(|x| = x\) in that case. If \(x\) is negative, on the other hand, then \(|x| = -x\), which is positive as well because the negative of a negative number is positive.

3. From the perspective of \(|x|\) being \(x\)’s distance from 0, a number and its opposite are the same distance from 0.
   
   From the function perspective, \(|x| = |−x|\) for every \(x\). For example, \(|2| = |-2| = 2\). If \(x > 0\), then \(-x < 0\), so \(|−x| = −(-x) = x = |x|\). If \(x < 0\), then \(|−x| = −x\) since \(-x > 0\). But, \(|x| = −x\) since \(x < 0\), so \(|x| = |−x|\).

Equations with Absolute Value

Suppose we want to solve the algebraic equation \(|x - 3| = 2\). There are two possibilities here: either \(x - 3 = 2\) or \(x - 3 = -2\). This is because both 2 and -2 have an absolute value of 2. If \(x - 3 = -2\), then \(|x - 3| = |-2| = 2\).

To phrase this equation another way, we wish to find \(x\) such that \(x - 3\) is 2 units away from 0. Thus, either \(x - 3 = 2\) or \(x - 3 = -2\) will work since both 2 and -2 are 2 units away from 0. We solve for \(x\) in each case:

\[
x - 3 = 2 \quad \text{or} \quad x - 3 = -2
\]

\[
x = 5 \quad \text{or} \quad x = 1.
\]

In general, if we have \(|f(x)| = c\), then we solve both \(f(x) = c\) and \(f(x) = -c\) to get possibly two values of \(x\). Let’s see two more examples:

**Example 1.** Solve \(|2x - 3| = 7\).

There are two cases: \(2x - 3 = 7\) and \(2x - 3 = -7\):

\[
2x - 3 = 7 \quad \text{or} \quad 2x - 3 = -7
\]

\[
2x = 10 \quad \text{or} \quad 2x = -4
\]

\[
x = 5 \quad \text{or} \quad x = -2.
\]
Example 2. Solve $\left| \frac{2}{3}x - 2 \right| = 1$.

Again, there are two cases: $\frac{2}{3}x - 2 = 1$ and $\frac{2}{3}x - 2 = -1$:

\[
\begin{align*}
\frac{2}{3} x - 2 &= 1 & \frac{2}{3} x - 2 &= -1 \\
\frac{2}{3} x &= 3 & \frac{2}{3} x &= 1 \\
x &= \frac{3}{2} (3) & x &= \frac{3}{2} (1) \\
x &= \frac{9}{2} & x &= \frac{3}{2}.
\end{align*}
\]

Inequalities with Absolute Value

Suppose we wish to solve the inequality $|x - 3| < 2$. As with the equalities, we will have two possibilities: either $x - 3 < 2$, or $x - 3 > -2$. Notice how in the second case, we switch the direction of the inequality in addition to changing 2 to $-2$.

We can also think of this inequality in terms of distance: we want to find $x$-values such that $x - 3$ is less than 2 units away from 0. Therefore, we wish for $x - 3$ to be between $-2$ and $2$. This gives us $-2 < x - 3 < 2$ (here we combined the two inequalities). Adding 3 to each expression gives $1 < x < 5$. We can show this result on a number line:

\[
\begin{array}{c}
\text{0} & \text{1} & \text{5} \\
(1, 5)
\end{array}
\]

The open circles represent the fact that the inequalities are strict so that $x \neq 1$ and $x \neq 5$. Let’s see a couple more examples:

Example 3. Solve $| -2x + 5 | < 10$.

We have two cases: $-2x + 5 < 10$ and $-2x + 5 > -10$. We can write this as a single chain of inequalities: $-10 < -2x + 5 < 10$, i.e. $-2x + 5$ is between $-10$ and 10. We now solve for $x$:

\[
\begin{align*}
-10 &< -2x + 5 < 10 \\
5 &> x - \frac{5}{2} > -5 & \text{Divide both sides by } -2.
\end{align*}
\]

\[
\begin{align*}
\frac{5}{2} &> x \quad \frac{5}{2} > -x & \text{Add } \frac{5}{2} \text{ to both sides.}
\end{align*}
\]

Going from the first line to the second line, remember that multiplying or dividing by a negative number switches the direction of the inequalities! We now graph our result $-\frac{5}{2} < x < \frac{15}{2}$:

\[
\begin{array}{c}
-\frac{5}{2} & 0 & \frac{15}{2} \\
\left( -\frac{5}{2}, \frac{15}{2} \right)
\end{array}
\]

Again, the circles are not filled in because the inequalities are strict.

Example 4. Solve $|4x - 2| \geq 6$.
Again, we have two cases: $4x - 2 \geq 6$ and $4x - 2 \leq -6$. Notice that here we cannot write a chain of inequalities since $4x - 2$ is not between $-6$ and $6$. We now solve both cases:

$$
\begin{align*}
4x - 2 &\geq 6 & 4x - 2 &\leq -6 \\
4x &\geq 8 & 4x &\leq -4 \\
x &\geq 2 & x &\leq -1.
\end{align*}
$$

We graph this solution below on a number line:

$$
\begin{array}{c}
-1 & 0 & 2 \\
\hline \\
\end{array}
$$

$[-1, 2]$

Notice that we have filled in the circles here, since the inequalities are now inclusive.