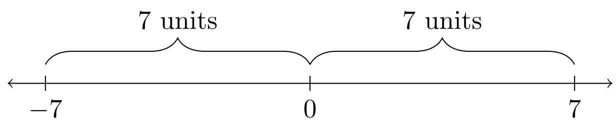
Absolute Value

Definition and Properties of Absolute Value

The **absolute value** of a number represents its distance from 0 on the number line. Below, we show -7 and 7 on a number line. By the function definition above, $|-7| = |7| = 7$, and in the picture we see that both -7 and 7 are 7 units away from 0.



We can also express the absolute value as a function, we define the **absolute value of x** as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \,. \end{cases}$$

The absolute value of a number satisfies certain properties, which we can verify with the function definition or which we can intuit by viewing the absolute value of a number as its distance from 0.

1. The distance from 0 to 0 on a number line is 0.

   Alternatively, $|0| = 0$ since plugging 0 into the function above yields 0.

2. The distance from any nonzero number to 0 on the number line is positive since nonzero distances are always positive (a negative distance doesn't make sense).

   Furthermore, if $a \neq 0$, then $|a| > 0$. By the function definition above, we see that $|x|$ is positive whenever $x > 0$ since $|x| = x$ in that case. If x is negative, on the other hand, then $|x| = -x$, which is positive as well because the negative of a negative number is positive.

3. From the perspective of $|x|$ being x's distance from 0, a number and its opposite are the same distance from 0.

   From the function perspective, $|x| = |-x|$ for every x. For example, $|2| = |-2| = 2$. If $x > 0$, then $-x < 0$, so $|-x| = -(-x) = x = |x|$. If $x < 0$, then $|-x| = -x$ since $-x > 0$. But, $|x| = -x$ since $x < 0$, so $|x| = |-x|$.

Equations with Absolute Value

Suppose we want to solve the algebraic equation $|x - 3| = 2$. There are two possiblities here: either $x - 3 = 2$ or $x - 3 = -2$. This is because both 2 and -2 have an absolute value of 2. If $x - 3 = -2$, then $|x - 3| = |-2| = 2$.

To phrase this equation another way, we wish to find x such that $x - 3$ is 2 *units away from* 0. Thus, either $x - 3 = 2$ or $x - 3 = -2$ will work since both 2 and -2 are 2 units away from 0. We solve for x in each case:

$$\begin{aligned} x - 3 &= 2 & \qquad x - 3 &= -2 \\ x &= 5 \quad \text{or} & x &= 1 \,. \end{aligned}$$

In general, if we have $|f(x)| = c$, then we solve both $f(x) = c$ and $f(x) = -c$ to get possibly two values of x. Let's see two more examples:

Example 1. Solve $|2x - 3| = 7$.

There are two cases: $2x - 3 = 7$ and $2x - 3 = -7$:

$$\begin{aligned} 2x - 3 &= 7 & \qquad 2x - 3 &= -7 \\ 2x &= 10 & 2x &= -4 \\ x &= 5 \quad \text{or} & x &= -2 \,. \end{aligned}$$

Example 2. Solve $\left|\frac{2}{3}x-2\right|=1$.

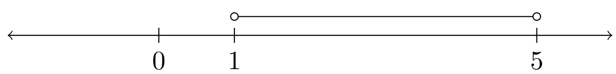
Again, there are two cases: $\frac{2}{3}x-2=1$ and $\frac{2}{3}x-2=-1$:

$$\begin{aligned} \frac{2}{3}x-2&=1 & \qquad \frac{2}{3}x-2&=-1 \\ \frac{2}{3}x&=3 & \frac{2}{3}x&=1 \\ x&=\frac{3}{2}(3) & x&=\frac{3}{2}(1) \\ x&=\frac{9}{2} \qquad \text{or} & x&=\frac{3}{2}\,. \end{aligned}$$

Inequalities with Absolute Value

Suppose we wish to solve the inequality $|x-3|<2$. As with the equalities, we will have two possibilities: either $x-3<2$, or $x-3>-2$. Notice how in the second case, we switch the direction of the inequality in addition to changing 2 to -2.

We can also think of this inequality in terms of distance: we want to find x-values such that $x-3$ is less than 2 units away from 0. Therefore, we wish for $x-3$ to be between -2 and 2. This gives us $-2<x-3<2$ (here we combined the two inequalities). Adding 3 to each expression gives $1<x<5$. We can show this result on a number line:



$(1,5)$

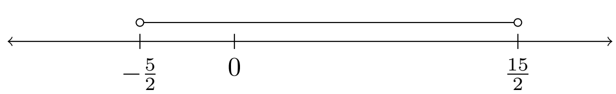
The open circles represent the fact that the inequalities are strict so that $x\neq 1$ and $x\neq 5$. Let's see a couple more examples:

Example 3. Solve $|-2x+5|<10$.

We have two cases: $-2x+5<10$ and $-2x+5>-10$. We can write this as a single chain of inequalities: $-10<-2x+5<10$, i.e. $-2x+5$ is between -10 and 10. We now solve for x:

$$\begin{aligned} -10 < -2x+5 &< 10 \\ 5 > \ x-\frac{5}{2} &> -5 \qquad \text{Divide both sides by } -2. \\ \frac{15}{2} > \quad x \quad &> -\frac{5}{2} \qquad \text{Add } \tfrac{5}{2} \text{ to both sides.} \end{aligned}$$

Going from the first line to the second line, remember that multiplying or dividing by a negative number switches the direction of the inequalities! We now graph our result $-\frac{5}{2}<x<\frac{15}{2}$:



$\left(-\frac{5}{2},\frac{15}{2}\right)$

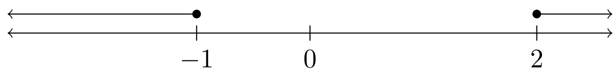
Again, the circles are not filled in because the inequalities are strict.

Example 4. Solve $|4x-2|\geq 6$.

Again, we have two cases: $4x - 2 \geq 6$ and $4x - 2 \leq -6$. Notice that here we cannot write a chain of inequalities since $4x - 2$ is not between -6 and 6. We now solve both cases:

$$\begin{aligned} 4x - 2 &\geq 6 & \qquad 4x - 2 &\leq -6 \\ 4x &\geq 8 & 4x &\leq -4 \\ x &\geq 2 \quad \text{or} & x &\leq -1\,. \end{aligned}$$

We graph this solution below on a number line:



$[-1, 2]$

Notice that we have filled in the circles here, since the inequalities are now inclusive.