Summary

When sketching curves, you use certain computable features of the function rather than plotting the graph point-by-point. Below, we give a table describing these various computable features, along with how to actually calculate them. "Function" refers to $f(x)$, and the two following columns refer to $f'(x)$ and $f''(x)$ respectively. When actually sketching a curve, the order you'll usually want to follow the order of the table below.

Feature	Function	First derivative	Second derivative
x-intercept	$f(x)$ at $x = 0$		
y-intercept	$f(x) = 0$		
Vertical asymptote	$f(x) \to \pm\infty$ as $x \to c$		
Horizontal asymptote	$f(x) \to c$ as $x \to \pm\infty$		
Increasing	$f(x)$ increases as x increases	$f'(x) > 0$	
Decreasing	$f(x)$ decreases as x increases	$f'(x) < 0$	
Relative extremum		$f'(x) = 0$ or is undefined.	
Relative maximum	$f(c) \geq f(x)$ around $x = c$	$f'(x)$ changes from positive to negative	$f''(x) < 0$ (and $f'(x) = 0$ or is undefined)
Relative minimum	$f(c) \leq f(x)$ around $x = c$	$f'(x)$ changes from negative to positive	$f''(x) > 0$ (and $f'(x) = 0$ or is undefined)
Concave up	$f(x)$ "cups" up	$f'(x)$ increasing	$f''(x) > 0$
Concave down	$f(x)$ "cups" down	$f'(x)$ decreasing	$f''(x) < 0$
Point of inflection	$f(x)$ changes concavity		$f''(x)$ changes sign.

Note that there's more than one way to find relative maxima and minima; you can determine them by looking at either the first derivative or the second derivative. When computing horizontal asymptotes, be sure to check $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$, as these two values might be different.

What is curve sketching?

One method for graphing a function $f(x)$ is to choose various x-values and then compute the corresponding function values at each x-value. From there, you try and fill in the gaps with some sort of curve. The problem with this method, is that it's often not clear *how* to draw this curve? After all, you could draw infinitely many curves between two points.

This handout on curve sketching addresses this issue. By finding asymptotes, intervals of increase and decrease, intervals of concavity, critical points, and points of inflection, you can adequately sketch most curves without computing a large number of points. Curve sketching brings together many of the concepts you learn in Differential Calculus, and so professors often test on this topic.

Asymptotes

You have likely seen three types of asymptotes: vertical, horizontal, and slant. We'll only cover the first two, as slant asymptotes appear relatively rarely. We'll define vertical and horizontal asymptotes and then give an example in which you find asymptotes of both types.

A function $f(x)$ has a **vertical asymptote** at $x = c$ if either $\lim_{x \to c^+} f(x)$ or $\lim_{x \to c^-} f(x)$ (or both) is $\pm\infty$. Vertical asymptotes come up most frequently when division by 0 is possible: when you're dealing with rational functions. In this case, you can determine where the function has vertical asymptotes by finding where the denomimator equals 0.

A function $f(x)$ has a **horizontal asymptote** if $\lim_{x \to \infty} f(x)$ and/or $\lim_{x \to -\infty} f(x)$ are finite numbers. Informally, you can find the horizontal asymptotes of functions by considering what happens as x gets "very large" (in absolute value). It is important to check the limits at both positive *and* negative infinity, as occasionally these limits can be different.

*Example 1.* Find the horizontal and vertical asymptotes of $f(x) = \dfrac{2x}{1-x}$.

First, let's find the vertical asymptotes. The denominator is 0 when $x = 1$, so there is a vertical asymptote there. Does $f(x)$ approach ∞ or $-\infty$ at $x = 1$? We compute

$$\lim_{x \to 1^-} f(x) = \infty \quad \text{and} \quad \lim_{x \to 1^+} f(x) = -\infty.$$

This information is helpful when sketching the curve: as x approaches 1 from the left, $f(x)$ will shoot upward, whereas x approaches 1 from the right, it will shoot downward. Next, let's consider the horizontal asymptotes. We calculate

$$\lim_{x \to -\infty} f(x) = -2, \quad \text{and} \quad \lim_{x \to \infty} f(x) = -2.$$

To sum up, there is one **vertical asymptote** at $x = 1$ and a **horizontal asymptote** at $y = -2$. As $x \to 1^-$, $f(x) \to \infty$, and as $x \to 1^+$, $f(x) \to -\infty$. Without computing any points, we have a pretty good idea of the shape of the function. We show its graph below.
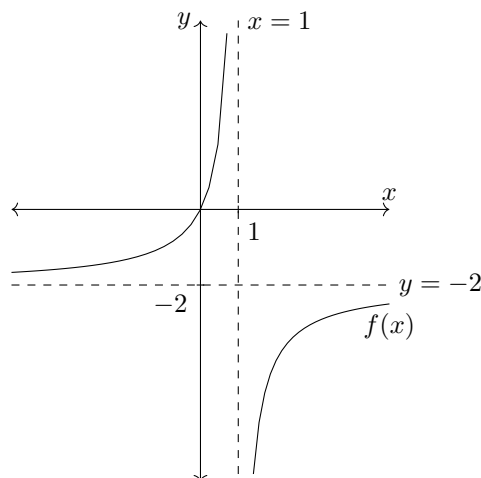
Intervals of increase and decrease and critical points

Informally, a graph is increasing if, when tracing the graph from left to right, the graph is going up, and a graph is decreasing if it goes down when tracing the graph from left to right. In quantitative terms, a function $f(x)$ is **increasing** when $f'(x)$, the slope of the tangent line to the curve, is positive, and $f(x)$ is **decreasing** when $f'(x)$ is negative. In the plot below, we plot $f(x) = x^2$ and its tangent lines. Left of the y-axis, the tangent lines have negative slope, and to the right of the y-axis, the tangent lines have positive slope. We see that $f(x)$ is decreasing on the interval $(-\infty, 0)$, and it is increasing on the interval $(0, \infty)$.
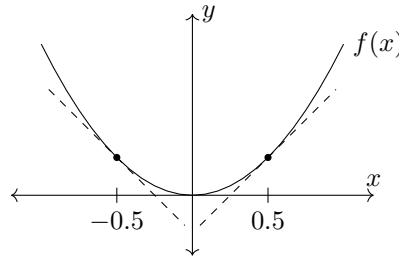
When $f'(x) = 0$, the $f(x)$ is neither increasing nor decreasing. Furthermore, $f(x)$ changes from increasing to decreasing (or vice versa) at values where $f'(x) = 0$ or where $f'(x)$ is undefined. Thus, in the intervals between these points, $f(x)$ is either only increasing or only decreasing. We determine intervals of increase and decrease by finding where $f'(x) = 0$ or is undefined. Such an x-value is called a **critical point** of $f(x)$.

There are two types of **relaitve extrema** in a graph: relative minima and relative maxmima. A function $f(x)$ has a **relative minimum** at $x = c$ if $f(c) \leq f(x)$ for all x-values "around" $x = c$. $f(x)$ has a **relative maximum** at $x = c$ if $f(c) \geq f(x)$ for all x-values "around" $x = c$. The *around* is important, and can be defined in a more precise sense, but we will illustrate it instead with an example. Before we go into the example, we also point out that a relative minimum occurs when $f(x)$ changes from decreasing to increasing, and a relative maximum occurs when $f(x)$ changes from increasing to decreasing.

Example 2. Find the intervals on which $f(x) = \frac{1}{3}x^3 + x^2 - 3x$ is increasing and on which it is decreasing.

We first compute
$$f'(x) = x^2 + 2x - 3 = (x+3)(x-1),$$

so $f'(x) = 0$ at $x = -3$ and at $x = 1$. These are our critical points. We now plug in points in each of the three intervals $(-\infty, -3), (-3, 1)$, and $(1, \infty)$:

x-values	$(x+3)$	$(x-1)$	$(x+3)(x-1)$
$x = -4$	$-$	$-$	$+$
$x = 0$	$+$	$-$	$-$
$x = 2$	$+$	$+$	$+$

The first x-value, $x = -4$, corresponds to the interval $(-\infty, -3)$, and so on. The rightmost column of our table tells us that $f'(x)$ is positive on the intervals $(-\infty, -3)$ and $(1, \infty)$ and is negative on $(-3, 1)$. Thus, the original function, $f(x)$, is **increasing** on $(-\infty, -3)$ and $(1, \infty)$–where $f'(x)$ is positive–and is **decreasing** on $(1, \infty)$–where $f'(x)$ is negative.

We can also determine where $f(x)$ has its relative extrema. We know they are at $x = -3$ and at $x = 1$. At $x = -3$, $f(x)$ changes from increasing to decreasing, so it has a **relative maximum** at $x = -3$. At $x = 1$, $f(x)$ changes from decreasing to increasing, so it has a **relative minimum** there. Notice how for x-values "around" $x = -3$, $f(-3)$ is the largest function value, even though as $x \to \infty$, the $f(x)$ surpasses this value. Similarly, "around" $x = 1$, $f(1)$ is the smallest function value, though $f(x)$ eventually gets lower than this as $x \to -\infty$.
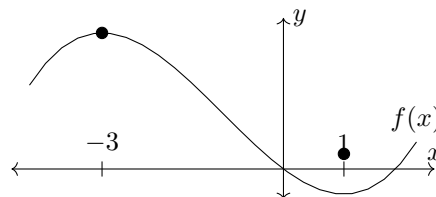
Concavity and points of inflection

Concavity describes the "cupping" of a function at a particular point or interval. When $f'(x)$ is increasing on an interval, the graph of $f(x)$ is **concave up**, and when $f'(x)$ is decreasing on an interval, the graph of $f(x)$ is **concave down**. Just as we used the first derivative to determine whether $f(x)$ was increasing or decreasing, we use the second derivative of $f(x)$ (the first derivative of $f'(x)$) to determine where $f'(x)$ is increasing or decreasing.

Thus, $f(x)$ is concave up when $f''(x) > 0$ ($f'(x)$ is increasing) and is concave down when $f''(x) < 0$ ($f'(x)$ is decreasing). A **point of inflection** is where $f(x)$ changes concavity, which must occur where $f'(x)$ changes from increasing to decreasing or vice versa. Let's revisit Example 2 to talk about concavity and points of inflection.

Example 2, continued. Find the intervals of concavity for $f(x) = \frac{1}{3}x^3 + x^2 - 3x$.

We already calculated $f'(x) = x^2 + 2x - 3$. We take the derivative again to get

$$f''(x) = 2x + 2.$$

$f''(x) = 0$ at $x = -1$. This divides our domain into two intervals: $(-\infty, -1)$ and $(-1, \infty)$. Plugging in $x = -2$, we get $f''(-2) = -2 < 0$, and plugging in $x = 0$ gives $f''(0) = 2 > 0$, so the original function $f(x)$ is **concave down** on $(-\infty, -1)$ and **concave up** on $(-1, \infty)$. Since $f(x)$ changes concavity at $x = -1$, it has a **point of inflection** at $x = -1$.

Second Derivative Test

Notice how $f(x)$ in Example 2 is concave down at its maximum at $x = -3$, and it's concave up at its minimum at $x = 1$. Recall that at maxima, $f'(x)$ changes from positive to negative. Thus, $f'(x)$ is decreasing at maxima. Furthermore, $f'(x)$ changes from negative to positive at minima, so $f'(x)$ is increasing at those points. The **Second Derivative Test** comes from this observation: if $f(x)$ has a critical point at $x = c$, then...

- if $f''(c) > 0$, then $f(x)$ has a relative minimum at $x = c$.

- if $f''(c) < 0$, then $f(x)$ has a relative maximum at $x = c$.

This is useful for classifying extreme points of $f(x)$ when you can compute $f''(x)$, and gives an alternative method to finding the intervals of increase and decrease.

Example 3. Sketch the curve for $f(x) = x^3 - 6x^2 + 9x + 1$.

First, we'll check for intercepts to this function. $f(0) = 1$, so $f(x)$ has a y-intercept at $(0, 1)$. At the moment, we can't really find the x-intercepts (when $f(x) = 0$), so we'll skip that part for now.

Next, we'll check for asymptotes. Since $f(x)$ is continuous for all x, it has no vertical asymptotes. Since $f(x)$ is unbounded as x approaches $\pm\infty$, there are no horizontal asymptotes either.

We now turn to intervals on which $f(x)$ is increasing and on which it is decreasing. To this end, we first compute $f'(x)$ and set it equal to 0:

$$f'(x) = 3x^2 - 12x + 9 = 3\left(x^2 - 4x + 3\right) = 3(x - 1)(x - 3) = 0.$$

From this, we see that $f'(x) = 0$ at $x = 1$ and $x = 3$. Furthermore, $f'(x)$ is defined everywhere, so our critical points are only at $x = 1$ and $x = 3$. $f(x)$ is increasing when $f'(x) > 0$, and it is decreasing when $f'(x) < 0$. To see where $f'(x)$ satisfies these inequalities, we use the following table:

x-values	$(x-1)$	$(x-3)$	$3(x-1)(x-3)$
$x = 0$	$-$	$-$	$+$
$x = 2$	$+$	$-$	$-$
$x = 4$	$+$	$+$	$+$

The roots split our number line into three intervals: $(-\infty, 1)$, $(1, 3)$, and $(3, \infty)$. Below, we graph each of these intervals, separating them with an open circle to represent where $f'(x) = 0$, and write the sign of $f'(x)$ on each interval.
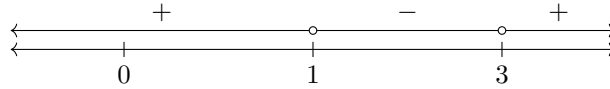
From this plot, we see that $f'(x)$ is increasing on the intervals $(-\infty, 1)$ and $(3, \infty)$ and that it is decreasing on $(1, 3)$.

To find relative extrema, we may refer back to this plot. Our relative extrema occur when $f'(x) = 0$ or $f'(x)$ is undefined, which occurs here at $x = 1$ and at $x = 3$. At $x = 1$, we see that $f'(x)$ changes from positive to negative, i.e. $f(x)$ changes from increasing to decreasing. This indicates that $f(x)$ has a relative maximum at $x = 1$. At $x = 3$, we see that $f'(x)$ changes from negative to positive, i.e. $f(x)$ changes from decreasing to increasing. This tells us that $f(x)$ has a relative minimum at $x = 3$.

For our final step, we'll look at concavity. We'll need the second derivative of $f(x)$:

$$f''(x) = 6x - 12\,.$$

This is 0 at $x = 2$. For $x < 2$, we see that $f''(x) < 0$, and for $x > 2$, we have $f''(x) > 0$. This shows that $f(x)$ is concave up on $(-\infty, 2)$ and concave down on $(2, \infty)$. There is a point of inflection at $x = 2$, since that's where $f''(x) = 0$. There are no points where $f''(x)$ is undefined. Before we sketch our graph, note that we could have used $f''(x)$ to classify the relative extrema of $f(x)$. At $x = 1$, $f''(x) < 0$, so $f(x)$ has a relative maximum, and at $x = 3$, $f''(x) > 0$, so $f(x)$ has a relative minimum.

We now sketch our graph. First, we'll plot our intercepts and critical points. We have a y-intercept at $(0, 1)$, a relative maximum at $x = 1$, and a relative minimum at $x = 3$. To find the y-coordinates of these two latter points, we could plug these x-values back into $f(x)$, though often this isn't necessary for the sketch. The fact that $f(x)$ has a relative maximum at $x = 1$ means that $f(x)$ must bend downwards at $x = 1$, and similarly $f(x)$ having a relative minimum at $x = 3$ means that it must bend up there.
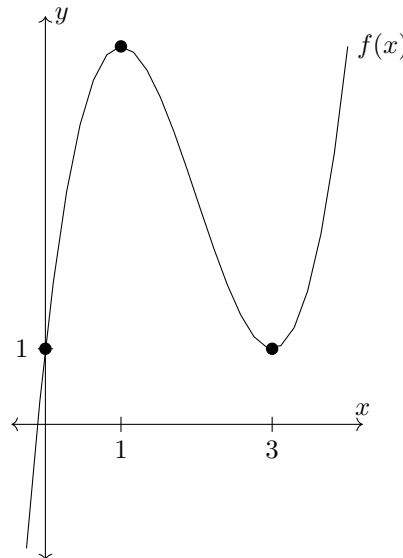
Notice how this sketch matches the intervals of increase and decrease and the intervals of concavity. Also notice how we only computed 3 points and then used the information about $f'(x)$ and $f''(x)$ to fill in the space between the points.