Eigenvalue-Eigenvector Method

What are eigenvectors and eigenvalues?

In the "Eigenvalues and Eigenvectors" handout, we discussed how to find the eigenvalues and eigenvectors of a matrix. As a reminder, an **eigenvector** of a matrix $\mathbf{A}$ is a vector, $\mathbf{v}^{(\lambda)}$, with an associated **eigenvalue**, λ satisfying

$$\mathbf{A}\mathbf{v}^{(\lambda)} = \lambda\mathbf{v}^{(\lambda)}.$$

In other words, the matrix $\mathbf{A}$ just acts on $\mathbf{v}^{(\lambda)}$ by multiplying the vector by a scalar. It is a theorem of Linear Algebra that eigenvectors for distinct eigenvalues are linearly independent. This will be important when we try to solve systems of differential equations.

The eigenvalue-eigenvector method, distinct roots

In general, we will use eigenvalues and eigenvectors to solve first-order, homogeneous, linear differential equations of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

where $\mathbf{A}$ is an $n \times n$ matrix, $\mathbf{x}$ is a vector with n components, and $\dot{\mathbf{x}} = \mathrm{d}\mathbf{x}/\mathrm{d}t$.

Example 1. Find a solution to the initial value problem, where

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

We first compute the characteristic polynomial of this matrix and set it equal to 0:

$$0 = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1),$$

which gives eigenvalues of -1 and 3. We now compute the eigenvectors for each eigenvalue.

- $\lambda = -1$: We wish to solve

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  Row-reduction gives

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

  which yields the single nontrivial equation $v_1 = -v_2$, corresponding to the eigenvector

$$\mathbf{v}^{(-1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- $\lambda = 3$: We wish to solve

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  Row-reduction gives

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \mathbf{v}^{(3)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

By the discussion above, each of these eigenvectors corresponds to a linearly independent solution to the differential equation. In particular,

$$\lambda = -1, \mathbf{v}^{(-1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \longrightarrow \mathbf{x}(t) = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\lambda = 3, \mathbf{v}^{(3)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow \mathbf{x}(t) = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This gives a general solution of

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The initial value condition gives

$$\begin{pmatrix} 0 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow c_1 = -1, c_2 = 1,$$

so that

$$\mathbf{x}(t) = -e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The eigenvalue-eigenvector method, complex roots

When the characteristic polynomial has complex (but distinct) roots, we use a very similar method as before. To illustrate the differences between the two methods, we give an example. In general, the complex roots always come in conjugate pairs (by elementary algebra), and for each pair we only need to get an eigenvector for one of the eigenvalues in that pair. From this eigenvalue and eigenvector we can get two linearly independent solutions, using the method we show below.

Example 2. Solve the system of differential equations given by

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \mathbf{x}.$$

We first compute the characteristic polynomial for this matrix:

$$\begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{vmatrix} = (2-\lambda)\left(\lambda^2+1\right).$$

This has eigenvalues $2, -i$, and i. We will find eigenvectors for 2 and i.

- $\lambda = 2$: We row-reduce to get:

$$\begin{pmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \longrightarrow \mathbf{v}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

- $\lambda = i$: We get the matrix

$$\begin{pmatrix} 1-i & 0 & 2 \\ 0 & 2-i & 0 \\ 1 & 0 & -1-i \end{pmatrix}.$$

Row-reducing complex matrices is often more complicated than row-reducing real matrices. As a general rule of thumb, we want to get complex numbers in one column and real numbers in another, so that adding rows together does not get new complex entries. Multiplying rows by the conjugate of a particular entry "shuffles" the complex entries around. For example, if we multiply the first row by $1+i$, we get a real entry in the first column and a complex entry in the third column. We can then cancel:

$$\begin{pmatrix} 1-i & 0 & 2 \\ 0 & 2-i & 0 \\ 1 & 0 & -1-i \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & -2-2i \\ 0 & 2-i & 0 \\ 1 & 0 & -1-i \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1+i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This gives $v_2 = 0$ and $v_1 = -(1+i)v_3$. This gives an eigenvector of

$$\mathbf{v}^{(i)} = \begin{pmatrix} -1-i \\ 0 \\ 1 \end{pmatrix}.$$

We could compute an eigenvector for $\lambda = -i$ and then plug that into our general solution. That would give us complex eigenvectors however, while it is possible to get real eigenvectors. We will use the fact that

$$e^{\alpha+\beta i} = e^{\alpha}\left(\cos(\beta) + i\sin(\beta)\right).$$

Now, we know that our solution for $\lambda = i$ is

$$\mathbf{x}(t) = e^{it}\begin{pmatrix} -1-i \\ 0 \\ 1 \end{pmatrix}.$$

To get two linearly independent, real solutions from this, we first expand e^{it}:

$$\mathbf{x}(t) = (\cos t + i\sin t)\begin{pmatrix} -1-i \\ 0 \\ 1 \end{pmatrix}.$$

Next, we break up the vector into its real and imaginary components:

$$\mathbf{x}(t) = (\cos t + i\sin t)\left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\right).$$

Now, we distribute:

$$\mathbf{x}(t) = \cos t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + i\cos t\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + i\sin t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \sin t\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

Lastly, we split this expression up into real and imaginary components:

$$\begin{aligned}\mathbf{x}(t) &= \left(\cos t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \sin t\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\right) + i\left(\cos t\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \sin t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} -\cos t + \sin t \\ 0 \\ \cos t \end{pmatrix} + i\begin{pmatrix} -\cos t - \sin t \\ 0 \\ \sin t \end{pmatrix}.\end{aligned}$$

Now, we have two real vectors: the real component and the imaginary component of the above expression.