Eigenvalues and Eigenvectors

What is an Eigenvector?

Consider the matrix $\mathbf{A}$ and vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We compute $\mathbf{Av}_1, \mathbf{Av}_2$, and $\mathbf{Av}_3$:

$$\mathbf{Av}_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$\mathbf{Av}_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\mathbf{Av}_3 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

The first product, $\mathbf{Av}_1$ isn't too special, but there is something interesting about $\mathbf{Av}_2$ and $\mathbf{Av}_3$. In each case, multiplying by $\mathbf{A}$ gives a scalar multiple of that vector. In particular, $\mathbf{Av}_2 = -\mathbf{v}_2$, and $\mathbf{Av}_3 = 3\mathbf{v}_3$.

A vector $\mathbf{v} \neq \mathbf{0}$ is an **eigenvector** for a matrix $\mathbf{A}$ if there is some scalar λ, called an **eigenvalue**, such that

$$\mathbf{Av} = \lambda\mathbf{v}.$$

Note that we do not count the zero vector, $\mathbf{0}$, as an eigenvector. We do, however, count 0 as a possible eigenvalue. Also, each eigenvalue λ can have two or more corresponding linearly independent eigenvectors. In our example above, $\mathbf{v}_2$ and $\mathbf{v}_3$ were eigenvectors of $\mathbf{A}$ with respective eigenvalues -1 and 3. Eigenvalues and eigenvectors have myriad applications in Linear Algebra, Differential Equations, and beyond. We will focus on finding the eigenvalues and eigenvectors from a matrix.

NOTE: It's better to think of the eigenvalues of a matrix $\mathbf{A}$ having associated eigenvectors rather than of the eigenvectors of a matrix having associated eigenvalues (as we defined it here). When using matrices to represent linear transformations, the matrix depends on the choice of basis. In such situations, the eigenvectors will depend on the basis used, but the eigenvalues won't.

Finding Eigenvalues

Here we explain the theory behind finding eigenvalues. If you're not interested in this, then you can skip ahead to the example. For a given matrix $\mathbf{A}$, each eigenvalue λ has a corresponding eigenvector $\mathbf{v}^{(\lambda)}$. Don't confuse this with exponentiation–it doesn't usually make sense to square a vector. The definition of an eigenvector and an eigenvalue gives

$$\mathbf{Av}^{(\lambda)} = \lambda\mathbf{v}^{(\lambda)} = \lambda\left(\mathbf{Iv}^{(\lambda)}\right),$$

where $\mathbf{I}$ is the identity matrix. Subtracting $\lambda\left(\mathbf{Iv}^{(\lambda)}\right)$ from both sides gives

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v}^{(\lambda)} = \mathbf{0}.$$

This equation and the fact that $\mathbf{v}^{(\lambda)}$ is a nonzero vector imply that the matrix $\mathbf{A} - \lambda\mathbf{I}$ is singular (or non-invertible), and so has a determinant of 0. Thus,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

is the condition we will use to find our eigenvalues. This determinant will give a polynomial in λ, which we call the **characteristic polynomial** of the matrix $\mathbf{A}$.

Example 1. Find the eigenvalues of the matrix $\mathbf{A}$ described in the introduction.

We must compute the determinant of $\mathbf{A} - \lambda\mathbf{I}$. Here, $\mathbf{I}$ is the 2×2 identity matrix. This gives

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}.$$

We now compute the determinant of this matrix:

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3\,.$$

To solve for λ, we set our characteristic polynomial $\lambda^2 - 2\lambda - 3$ equal to 0. We then solve for λ via factoring:

$$0 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)\,.$$

This gives us our eigenvalues of $\lambda = -1$ and $\lambda = 3$.

Finding Eigenvectors

Once we have found our eigenvalues, we want to solve

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}^{(\lambda)} = \mathbf{0}$$

for each eigenvalue λ. In our example, we must do this for both $\lambda = -1$ and $\lambda = 3$.

Example 1, continued. Find the eigenvectors of the matrix $\mathbf{A}$ described in the introduction.

So far we've found that $\mathbf{A}$ has two eigenvalues: $\lambda = -1$ and $\lambda = 3$. For each eigenvalue, we must find one or more corresponding eigenvectors.

- $\lambda = -1$: The matrix equation we must solve is:

  $$(\mathbf{A} + \mathbf{I})\mathbf{v}^{(-1)} = \mathbf{0} \longrightarrow \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  Row-reduction (subtracting Row 1 from Row 2 and then scaling Row 1 by $\frac{1}{2}$) gives

  $$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

  which gives the single (nontrivial) equation $v_1 = -v_2$. Note that we may choose any value we wish for v_1. For simplicity, we set $v_1 = 1$, giving $v_2 = -1$. Notice how this vector was exactly the vector $\mathbf{v_2}$ in the introduction that had eigenvalue -1. Since this equation constrains both v_1 and v_2, we can't find any other linearly independent vectors.

- $\lambda = 3$: We must solve

  $$(\mathbf{A} - 3\mathbf{I})\mathbf{v}^{(-1)} = \mathbf{0} \longrightarrow \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  Row-reduction (adding Row 1 to Row 2 and then scaling Row 1 by $-\frac{1}{2}$) gives

  $$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

  which gives the single (nontrivial) equation $v_1 = v_2$. As before, we can choose whatever value we like for v_1, so we choose $v_1 = v_2 = 1$. Again, the equation constrains both v_1 and v_2, so we can't find any more linearly independent vectors. Notice how this is the vector $\mathbf{v}_3$ from the introduction, which had eigenvalue 3.

To conclude, we have found that our matrix $\mathbf{A}$ has eigenvalues -1 and 3, with corresponding eigenvectors

$$\mathbf{v}^{(-1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}^{(3)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Another Example

Example 2. Find the eigenvalues and eigenvectors of the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

We compute the characteristic polynomial of this matrix:

$$0 = \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & 4 & 1 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = -\lambda(1-\lambda)(2-\lambda) - 2(2-\lambda)\,,$$

which simplifies to

$$\begin{aligned} 0 = (2-\lambda)(-\lambda(1-\lambda)-2) &= (2-\lambda)(-\lambda+\lambda^2-2) \\ &= (2-\lambda)(\lambda-2)(\lambda+1) \\ &= -(\lambda-2)^2(\lambda+1)\,. \end{aligned}$$

$\lambda = -1$ and $\lambda = 2$ both solve this, so they are our eigenvalues.

- $\lambda = -1$: We must solve

$$(\mathbf{A} + \mathbf{I})\mathbf{v}^{(-1)} = \mathbf{0} \longrightarrow \begin{pmatrix} 2 & 4 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

  We can row-reduce this (first try subtracting Row 3 from Row 1):

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

  which gives $v_2 = 0$ and $2v_1 = -v_3$. We can get an eigenvector by setting $v_1 = 1, v_2 = 0$, and $v_3 = -2$.

- $\lambda = 2$: We must solve

$$(\mathbf{A} - 2\mathbf{I})\mathbf{v}^{(2)} = \mathbf{0} \longrightarrow \begin{pmatrix} -1 & 4 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

  We can row-reduce this (Scale Row 3 by $\frac{1}{2}$ and then add it to Row 1) to get

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

  This gives $v_2 = 0$ and $v_1 = v_3$, so we choose $v_1 = 1$ and $v_3 = 1$.

To sum up, we have eigenvalues -1 and 2 with corresponding eigenvectors

$$\mathbf{v}^{(-1)} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Note how even though the matrix is a 3×3 matrix, we have only two eigenvalues and two linearly independent vectors.