

Introduction

In our "Limits and Continuity" handout, we define the limit of a function $f(x)$ at $x = c$, $\lim_{x \rightarrow c} f(x)$. In determining the limits for a particular function $f(x)$, we looked at its graph. In general, this method is inconvenient. Furthermore, the formal delta-epsilon definition of the limit is unwieldy to use. Here, we will discuss a few *ad hoc* yet useful methods to determine the limit of the function from itself rather than from the graph.

Plugging in $x = c$

Suppose we wanted to find the limit as $x \rightarrow \frac{\pi}{2}$ of $\sin x$. Well, $\sin\left(\frac{\pi}{2}\right) = 1$. We are also aware that the graph of $\sin x$ behaves rather nicely at $x = \frac{\pi}{2}$. In particular, it is continuous. As discussed in the "Limits and Continuity" handout, for a continuous function $f(x)$, we have $\lim_{x \rightarrow c} f(x) = f(c)$, so

$$\lim_{x \rightarrow \frac{\pi}{2}} \sin x = \sin\left(\frac{\pi}{2}\right) = 1.$$

We can compute the following limits similarly:

$$\begin{aligned} \lim_{x \rightarrow 2} (x + 2) &= 2 + 2 = 4 \\ \lim_{x \rightarrow 3\pi} \cos x &= \cos(3\pi) = -1 \\ \lim_{x \rightarrow \ln 2} e^x &= e^{\ln 2} = 2. \end{aligned}$$

The functions $x + 2$, $\cos x$, and e^x are all continuous, which is why this approach works. Note that we could not use this method if we actually wanted to *show* that any of these functions is continuous since this method is based on the (true) *assumption* that these functions are continuous. We use our knowledge that certain functions already are continuous to evaluate their limits more easily.

The following are examples of common functions that are continuous (on their domains):

Type	Examples
Polynomials	1; $x + 2$; $x^3 - 4x + 1$
Trigonometric functions	$\sin x$; $\cos x$; $\tan x$ (if $\cos x \neq 0$)
Exponential functions	e^x ; 2^{-x}
Sums of cont. functions	$x^2 + \cos x$; $e^x - \sin x$
Products of cont. functions	$x(e^x + \cos x)$; $x^2 \cos x$
Quotients of cont. (denominator $\neq 0$)	$\frac{x^2}{x+1}$, ($x \neq -1$); $\frac{2}{e^x}$ (for all x)
Compositions of cont. functions	$\cos(x^2)$; $e^{\sin(x+2)}$

You can evaluate the limits of these functions at $x = c$ simply by plugging in c for x , so long as the function is defined at $x = c$. Since this method is one of the easier ones to use, it's good to try it first, even if it does not end up solving the problem.

Simplifying Rational Functions

Often, the above method fails because it results in division by 0, in particular with rational functions. Consider, for example, the limit

$$\lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{x^2 - x - 2}.$$

If we try to evaluate the function at $x = 2$, we get 0 in the denominator and in the numerator. We could evaluate this particular limit using L'Hôpital's Rule (discussed in another handout), but we can also handle

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this via factoring. Factoring the numerator and the denominator gives

$$\lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{x^2 - x - 2} = \lim_{x \rightarrow 2} \frac{(x-2)(x-2)}{(x-2)(x+1)}.$$

Since we're only taking the limit as $x \rightarrow 2$, we have $x - 2 \neq 0$ for our purposes (recall that we don't care what happens *at* $x = 2$, only around it). The evaluation (plugging-in) method we previously discussed *does* use the function value *at* $x = c$, so it would not work here. Since we are not using the evaluation method, we can treat $x - 2$ as not equal to 0. Hence, we can cancel out an $x - 2$ from the numerator and the denominator:

$$\lim_{x \rightarrow 2} \frac{(x-2)(x-2)}{(x-2)(x+1)} = \lim_{x \rightarrow 2} \frac{\cancel{(x-2)}(x-2)}{\cancel{(x-2)}(x+1)} = \lim_{x \rightarrow 2} \frac{x-2}{x+1}.$$

Now, this function *is* defined at $x = 2$. In fact it is continuous at $x = 2$ since the numerator and the denominator are continuous functions and $x + 1 \neq 0$ at $x = 2$. We can now use the evaluation method:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{x^2 - x - 2} = \lim_{x \rightarrow 2} \frac{x-2}{x+1} = \frac{2-2}{2+1} = 0.$$

We use this method whenever we can factor the numerator or the denominator.

Limits with ∞

Consider the limit

$$\lim_{x \rightarrow \infty} \frac{2x}{x^2 - 2x}.$$

Using the method described above, we can factor the numerator:

$$\lim_{x \rightarrow \infty} \frac{2x}{x(x-2)} = \lim_{x \rightarrow \infty} \frac{2}{x-2}.$$

As $x \rightarrow \infty$, the denominator $x - 2$ grows in size. The numerator, on the other hand, is fixed at 2. As x grows larger, we are dividing 2 by a larger and larger number, thus the fraction approaches 0. To verify this, divide 2 by 10, then by 1,000, then by 100,000, and notice how the result gets closer and closer to 0. In other words,

$$\lim_{x \rightarrow \infty} \frac{2}{x-2} = 0.$$

Now consider the limit

$$\lim_{x \rightarrow 0} \frac{5}{4x^2}.$$

As $x \rightarrow 0$, the denominator approaches 0. The numerator is fixed at 5. As x approaches 0, we are dividing 5 by a number approaching zero, so the entire fraction is growing. To check this, try dividing 5 by 0.1, then 0.01, and then 0.001, and note how the result grows without bound. The fraction grows without limit, so we have

$$\lim_{x \rightarrow 0} \frac{5}{4x^2} = \infty.$$

Keep in mind that we aren't dividing by 0, since we aren't trying to evaluate the function *at* $x = 0$. Furthermore, we would usually have to check the limit from the left and the right (more on that in a latter section). Here, the x -term is squared, so it makes no difference whether x is positive or negative.

Limits as $x \rightarrow \pm\infty$

In this section, we'll primarily focus on rational functions of polynomials. First, let's consider

$$\lim_{x \rightarrow \infty} \frac{2x^3 + x - 11}{-4x^3 - 2x - 1}.$$

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The numerator approaches ∞ and the denominator approaches $-\infty$, so it's not quite clear what the result will be. This form, $\frac{\infty}{\infty}$, is an example of an **indeterminate form**, which are covered in more detail in the L'Hôpital's Rule handout. We can handle this limit without L'Hôpital's Rule, however. First, look at the degrees (highest powers) of both the polynomials: in both the numerator and in the denominator, the degree is 3. We now multiply the numerator and denominator by a special form of 1:

$$\lim_{x \rightarrow \infty} \left(\frac{2x^3 + x - 11}{-4x^3 - 2x - 1} \cdot \frac{1/x^3}{1/x^3} \right) = \lim_{x \rightarrow \infty} \frac{2 + 1/x^2 - 11/x^3}{-4 - 2/x^2 - 1/x^3}.$$

In the numerator, the terms over x^2 and over x both approach 0 as $x \rightarrow \infty$, as discussed in the previous section. The same holds true of the last two terms in the denominator. We thus get

$$\lim_{x \rightarrow \infty} \frac{2 + 1/x^2 - 11/x^3}{-4 - 2/x^2 - 1/x^3} = \frac{2 + 0 - 0}{-4 - 0 - 0} = -\frac{1}{2}.$$

Let's try another example, this time when the numerator and the denominator have different degrees:

$$\lim_{x \rightarrow \infty} \frac{x^3 - x - 1}{2x^2 - 3}.$$

The numerator has a degree of 3, whereas the denominator has a degree of 2. Before, we multiplied by $\frac{1}{x^3}$ since that matched the degree of both the numerator and the denominator. We actually only have to match one of the degrees, however. Let's show both ways:

1. *Multiplying by $1/x^3$* : We get

$$\lim_{x \rightarrow \infty} \frac{x^3 - x - 1}{2x^2 - 3} \cdot \frac{1/x^3}{1/x^3} = \lim_{x \rightarrow \infty} \frac{1 - 1/x^2 - 1/x^3}{2/x - 3/x^3}.$$

As $x \rightarrow \infty$, the terms over powers of x all go to 0. This leaves a 1 in the numerator and something approaching 0 in the denominator. By the previous section, the limit is ∞ .

2. *Multiplying by $1/x^2$* : We get

$$\lim_{x \rightarrow \infty} \frac{x^3 - x - 1}{2x^2 - 3} \cdot \frac{1/x^2}{1/x^2} = \lim_{x \rightarrow \infty} \frac{x - 1/x - 1/x^2}{2 - 3/x}.$$

The $3/x$ in the denominator approaches 0 as $x \rightarrow \infty$. The numerator, on the other hand, still has an x -term, so as $x \rightarrow \infty$ the numerator blows up. Again, this gives us a limit of ∞ .

Let's try one more limit, this time with the denominator having a higher degree:

$$\lim_{x \rightarrow \infty} \frac{x + 3}{3x^2 - x - 1}.$$

Again, we can multiply the numerator and the denominator by either $1/x$ or $1/x^2$. Let's do the former:

$$\lim_{x \rightarrow \infty} \frac{x + 3}{3x^2 - x - 1} \cdot \frac{1/x}{1/x} = \lim_{x \rightarrow \infty} \frac{1 + 3/x}{3x - 1 - 1/x}.$$

As $x \rightarrow \infty$, the numerator approaches 1. The denominator, on the other hand, has an x -term, so it approaches ∞ . Thus, the fraction as a whole approaches 0.

In general, the process is:

1. Identify the degree of the numerator and of the denominator.
2. Multiply the numerator and denominator by $1/x^n$, where n is the degree of either the numerator or the denominator (it does not matter which).

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3. Evaluate this new quantity, keeping in mind that $\lim_{x \rightarrow \infty} 1/x = 0$.

This will result in three cases: (1) if the numerator has a higher degree, the limit is $\pm\infty$, (2) if the denominator has a higher degree, the limit is 0, and (3) if the degrees of the numerator and denominator are equal, then the limit is the ratio of the leading coefficients.

Limits with Absolute Value

First, we must give the proper definition for absolute value:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

This always gives a positive number. This is because if $x < 0$, then $-x > 0$, i.e. is positive. If we shift the argument (inside) of the absolute value, this changes where we decide to negate the argument:

$$|x - 2| = \begin{cases} x - 2 & x \geq 2 \\ 2 - x & x < 2. \end{cases}$$

To see why we use this function definition, we rewrite it in the first form:

$$|x - 2| = \begin{cases} x - 2 & x - 2 \geq 0 \\ -(x - 2) & x - 2 < 0 \end{cases} = \begin{cases} x - 2 & x \geq 2 \\ 2 - x & x < 2. \end{cases}$$

Now, consider the limit

$$\lim_{x \rightarrow 1} \frac{|x - 1|}{x - 1}.$$

Both the numerator and denominator are 0 at $x = 1$, so we can't just plug in $x = 1$. We can't cancel the $x - 1$ terms as we would in a polynomial, since the absolute value signs do not behave exactly like parentheses. We must consider left- and right-hand limits:

- As $x \rightarrow 1^-$ (approaches 1 from the left), $x < 1$, so the absolute value sign *negates* the argument:

$$\lim_{x \rightarrow 1^-} \frac{|x - 1|}{x - 1} = \lim_{x \rightarrow 1^-} \frac{-(x - 1)}{x - 1} = -1.$$

- As $x \rightarrow 1^+$ (approaches 1 from the right), $x > 1$, so the absolute value sign does nothing to the argument:

$$\lim_{x \rightarrow 1^+} \frac{|x - 1|}{x - 1} = \lim_{x \rightarrow 1^+} \frac{x - 1}{x - 1} = 1.$$

The left- and right-hand limits do not agree, so the limit itself does not exist. Consider now the limits

$$\lim_{x \rightarrow 2} \frac{|x - 1|}{x - 1} \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{|x - 1|}{x - 1}.$$

Here, we can just evaluate at $x = 2$ and at $x = 0$ respectively since this does not give us division by 0. The limits are 1 and -1 respectively. These two examples are just to show that we only need to check the left- and right-hand limits at the x -value where the argument equals 0 because that's where the absolute value sign changes behavior.

One-sided Limits

We now consider another case of taking limits where we get division by zero. Consider the limit

$$\lim_{x \rightarrow 1} \frac{-2x}{x - 1},$$

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which gives division by zero if we try to evaluate at $x = 1$. The numerator approaches -2 as $x \rightarrow 1$, so this might suggest that we have a limit of $-\infty$ since we are dividing a negative constant by a smaller and smaller number. We must be careful, however, because the denominator approaches 0 differently from the left and from the right:

- As $x \rightarrow 1^-$ (from the left), the denominator is negative, so we have a negative constant divided by a shrinking (in absolute value) *negative* number, which gives us a limit of ∞ .
- As $x \rightarrow 1^+$ (from the right), the denominator is positive, so we have a negative constant divided by a shrinking (in absolute value) *positive* number, which gives us a limit of $-\infty$.

The left- and right-hand limits do not agree, so the limit at $x = 1$ does not exist. On the other hand, consider

$$\lim_{x \rightarrow 1} \frac{-2x}{(x-1)^2}.$$

Now, from either side, the denominator is always positive (due to us squaring $x-1$), so we have our left- and right-hand limits are both $-\infty$, which gives a limit of $-\infty$.

Note that when $x \rightarrow \infty$ or $x \rightarrow -\infty$, we do not need to consider one-sided limits.

Limits with Radicals: Using the Conjugate

Suppose we want to evaluate the limit

$$\lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2}.$$

As $x \rightarrow 4$, the numerator and the denominator both approach 0 . To handle this example, we first take the conjugate of the denominator. The conjugate of $\sqrt{x}-2$ is $\sqrt{x}+2$. Next, we multiply the numerator and denominator by this conjugate:

$$\lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x \rightarrow 4} \frac{(x-4)(\sqrt{x}+2)}{x-4} = \lim_{x \rightarrow 4} \sqrt{x}+2 = 4.$$

Let's use this method for the limit

$$\lim_{x \rightarrow 1} \frac{\sqrt{x+3}-2}{x-1}.$$

Again, notice that the numerator and denominator both approach $z = 0$. This time, the radical is in the numerator. Nevertheless, we multiply both the numerator and the denominator by the conjugate:

$$\lim_{x \rightarrow 1} \frac{\sqrt{x+3}-2}{x-1} \cdot \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} = \lim_{x \rightarrow 1} \frac{(x+3)-4}{(x-1)(\sqrt{x+3}+2)}.$$

We could distribute out the term on the bottom, but it is usually helpful to simplify other things first. Here, the numerator simplifies to $x-1$. This gives

$$\lim_{x \rightarrow 1} \frac{x-1}{(x-1)(\sqrt{x+3}+2)} = \lim_{x \rightarrow 1} \frac{1}{\sqrt{x+3}+2} = \frac{1}{4}.$$

We'll see another use of multiplication by conjugates in this example:

$$\lim_{x \rightarrow \infty} \sqrt{x^2-3x} - \sqrt{x^2+3}.$$

The conjugate of the expression in the limit is $\sqrt{x-2} + \sqrt{x^2+2}$. We multiply this expression by the conjugate over itself:

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\sqrt{x^2-3x} - \sqrt{x^2+3} \right) &\cdot \frac{\sqrt{x^2-3x} + \sqrt{x^2+3}}{\sqrt{x^2-3x} + \sqrt{x^2+3}} = \lim_{x \rightarrow \infty} \frac{(x^2-3x) - (x^2+3)}{\sqrt{x^2-3x} + \sqrt{x^2+3}} \\ &= \lim_{x \rightarrow \infty} \frac{-3x-3}{\sqrt{x^2-3x} + \sqrt{x^2+3}}. \end{aligned}$$

Evaluating Limits

This is now of the form $\frac{\infty}{\infty}$, so it's still not quite clear what this limit is. As was the case with rational functions of polynomials, we look at the degree of the numerator and multiply both the numerator and the denominator by $\frac{1}{x}$:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{-3x - 3}{\sqrt{x^2 - 3x} + \sqrt{x^2 + 3}} \cdot \frac{1/x}{1/x} &= \lim_{x \rightarrow \infty} \frac{-3 - 3/x}{\sqrt{1 - 3/x} + \sqrt{1 + 3/x^2}} \\ &= \frac{-3}{1 + 1} \\ &= -\frac{3}{2}.\end{aligned}$$