First-Order Equations: 
Exact Equations

In this handout, our equations will be of the form

\[ M(t, y) + N(t, y)y' = 0. \]

The above equation is **exact** if the following condition holds:

\[ M_y(t, y) = N_t(t, y), \]

where \( M_y \) denotes the partial derivative of \( M \) with respect to \( y \), and \( N_t \) denotes the partial derivative of \( N \) with respect to \( t \). We choose to show that method for dealing with exact equations through examples.

**Example 1.** Solve \( 6y + 3t^2 + 6ty' = 0 \).

For this first example, we’ll touch on the theory behind the method for solving exact equations. Let’s identify \( M(t, y) \) and \( N(t, y) \) first. \( N(t, y) \) is the part that is multiplied by \( y' \), so here \( N(t, y) = 6t \). \( M(t, y) \) is the rest of the equation, \( 6y + 3t^2 \). To check whether this equation is exact, we compute

\[
M_y(t, y) = \frac{d}{dy}[6y + 3t^2] = 6, \\
N_t(t, y) = \frac{d}{dt}[6t] = 6.
\]

The differential equation is indeed exact. Now, let’s suppose that there is a function \( \psi(t, y) \) such that

\[
\psi_t(t, y) = M(t, y), \quad \text{and} \quad \psi_y(t, y) = N(t, y).
\]

In fact, such a \( \psi \) exists whenever the differential equation is exact, but we won’t discuss why. Then, by writing \( y = y(t) \) and \( \psi(t, y) = \psi(t, y(t)) \), the Chain Rule for partial derivatives gives us

\[
\psi_t + \psi_y y' = 0, \\
\frac{d}{dt} [\psi(t, y(t))] = 0, \\
\psi(t, y) = c,
\]

for some constant \( c \). Our goal for solving exact equations is to find this function \( \psi(t, y) \). We now turn back to our original problem, which we have shown was an exact equation. Then by the discussion above, we assume

\[
\psi_t = M = 6y + 3t^2, \quad \text{and} \quad \psi_y = N = 6t,
\]

which allows us to solve for \( \psi \). We compute

\[
\psi = \int M \, dt = \int (6y + 3t^2) \, dt = 6ty + t^3 + h(y),
\]

where \( h(y) \) is some function of \( y \). The \( h(y) \) acts as the “constant of integration” when we integrate with respect to \( t \) since the partial derivative of \( h(y) \) with respect to \( t \) is 0. Taking the derivative of both sides with respect to \( y \) gives

\[
\psi_y = 6t + h'(y).
\]

Recall that \( \psi_y = N = 6t \), so this implies that \( h'(y) = 0 \), so \( h(y) \) is a constant. We conclude that \( \psi(t, y) = 6ty + t^3 \). We write our general solution as follows:

\[
\psi(t, y) = 6ty + t^3 = c,
\]

where \( c \) is an arbitrary constant. What happened to \( h(y) \)? Since \( h(y) \) is a constant, we can move it to the right-hand side with the other constant, and the result is again a constant. This is an **implicit solution** for \( y \). We do not have an explicit “\( y = \)” equation, but we could use this implicit solution to approximate \( y \).
itself. In fact, we could isolate \( y \) algebraically here if we wished, though we cannot always do this.

**Example 2.** Solve \( 2t^2y\cos(t^2) + 2y\sin(t^2)y' = 0 \).

We first identify \( M \) and \( N \). We see that

\[
M(t, y) = 2t^2y\cos(t^2), \quad \text{and} \quad N(t, y) = 2y\sin(t^2).
\]

Next, we verify that the equation is exact by checking whether \( M_y = N_t \). To this end, we compute

\[
M_y = [2t^2y\cos(t^2)]_y = 4ty\cos(t^2), \quad N_t = [2y\sin(t^2)]_t = 4ty\cos(t^2),
\]

so the equation is exact. We then calculate

\[
\psi = \int M \, dt = \int 2t^2\cos(t^2) \, dt = y^2\sin(t^2) + h(y).
\]

As before, we must determine \( h(y) \), which we can do by examining \( \psi_y = 2y\sin(t^2) + h'(y) \) and using the fact that \( \psi_y = N = 2y\sin(t^2) \) to conclude that \( h'(y) = 0 \), so \( h(y) \) is a constant. This gives us a final answer of

\[
\psi(t, y) = y^2\sin(t^2) = c.
\]

Alternatively, we could compute \( \psi = \int N \, dy = y^2\sin(t^2) + g(t) \), where \( g(t) \) is solely a function of \( t \). Then we have

\[
y^2\sin(t^2) + h(y) = y^2\sin(t^2) + g(t),
\]

from which we conclude that \( g(t) = h(y) = 0 \).

**DISCLAIMER:** This handout uses notation and methods from the textbook commonly used for M 427J courses taught at the University of Austin:

Braun, Martin, *Differential Equations and Their Applications*, 4th ed. Springer

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