Exact Equations

In this handout, our equations will be of the form

$$M(t,y) + N(t,y)y' = 0\,.$$

The above equation is **exact** if the following condition holds:

$$M_y(t,y) = N_t(t,y)\,,$$

where M_y denotes the partial derivative of M with respect to y, and N_t denotes the partial derivative of N with respect to t. We choose to show that method for dealing with exact equations through examples.

Example 1. Solve $6y + 3t^2 + 6ty' = 0$.

For this first example, we'll touch on the theory behind the method for solving exact equations. Let's identify $M(t,y)$ and $N(t,y)$ first. $N(t,y)$ is the part that is multiplied by y', so here $N(t,y) = 6t$. $M(t,y)$ is the rest of the equation, $6y + 3t^2$. To check whether this equation is exact, we compute

$$\begin{array}{ccc} M_y(t,y) = \left[6y + 3t^2\right]_y & & N_t(t,y) = [6t]_t \\ 6 & = & 6\,. \end{array}$$

The differential equation is indeed exact. Now, let's suppose that there is a function $\psi(t,y)$ such that

$$\psi_t(t,y) = M(t,y)\,, \quad \text{and} \quad \psi_y(t,y) = N(t,y)\,.$$

In fact, such a ψ exists whenever the differential equation is exact, but we won't discuss why. Then, by writing $y = y(t)$ and $\psi(t,y) = \psi(t,y(t))$, the Chain Rule for partial derivatives gives us

$$\begin{aligned} \psi_t + \psi_y y' &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\psi(t,y(t))\right] &= 0 \\ \psi(t,y) &= c\,, \end{aligned}$$

for some constant c. Our goal for solving exact equations is to find this function $\psi(t,y)$. We now turn back to our original problem, which we have shown was an exact equation. Then by the discussion above, we assume

$$\psi_t = M = 6y + 3t^2\,, \quad \text{and} \quad \psi_y = N = 6t\,,$$

which allows us to solve for ψ. We compute

$$\psi = \int M\,\mathrm{d}t = \int 6y + 3t^2\,\mathrm{d}t = 6ty + t^3 + h(y)\,,$$

where $h(y)$ is some function of y. The $h(y)$ acts as the "constant of integration" when we integrate with respect to t since the partial derivative of $h(y)$ with respect to t is 0. Taking the derivative of both sides with respect to y gives

$$\psi_y = 6t + h'(y)\,.$$

Recall that $\psi_y = N = 6t$, so this implies that $h'(y) = 0$, so $h(y)$ is a constant. We conclude that $\psi(t,y) = 6ty + t^3$. We write our general solution as follows:

$$\psi(t,y) = 6ty + t^3 = c\,,$$

where c is an arbitrary constant. What happened to $h(y)$? Since $h(y)$ is a constant, we can move it to the right-hand side with the other constant, and the result is again a constant. This is an **implicit solution** for y. We do not have an explicit "$y =$" equation, but we could use this implicit solution to approximate y

itself. In fact, we could isolate y algebraically here if we wished, though we cannot always do this.

Example 2. Solve $2ty^2\cos\left(t^2\right) + 2y\sin\left(t^2\right)y' = 0$.

We first identify M and N. We see that

$$M(t,y) = 2ty^2\cos\left(t^2\right)\,, \quad \text{and} \quad N(t,y) = 2y\sin\left(t^2\right)\,.$$

Next, we verify that the equation is exact by checking whether $M_y = N_t$. To this end, we compute

$$\begin{array}{ccc} M_y = \left[2ty^2\cos\left(t^2\right)\right]_y & & N_t = \left[2y\sin\left(t^2\right)\right]_t \\ 4ty\cos\left(t^2\right) & = & 4ty\cos\left(t^2\right)\,, \end{array}$$

so the equation is exact. We then calculate

$$\psi = \int M\,\mathrm{d}t = \int 2ty\cos\left(t^2\right)\,\mathrm{d}t = y^2\sin\left(t^2\right) + h(y)\,.$$

As before, we must determine $h(y)$, which we can do by examining $\psi_y = 2y\sin\left(t^2\right) + h'(y)$ and using the fact that $\psi_y = N = 2y\sin\left(t^2\right)$ to conclude that $h'(y) = 0$, so $h(y)$ is a constant. This gives us a final answer of

$$\psi(t,y) = y^2\sin\left(t^2\right) = c\,.$$

Alternatively, we could compute $\psi = \int N\,\mathrm{d}y = y^2\sin\left(t^2\right) + g(t)$, where $g(t)$ is solely a function of t. Then we have

$$y^2\sin\left(t^2\right) + h(y) = y^2\sin\left(t^2\right) + g(t)\,,$$

from which we conclude that $g(t) = h(y) = 0$.

DISCLAIMER: This handout uses notation and methods from the textbook commonly used for M 427J courses taught at the University of Austin:
Braun, Martin, *Differential Equations and Their Applications*, 4^{th} ed. Springer
December 5, 1992.