Exponent and Log Laws

Introduction

In this handout, we’ll discuss exponent laws and logarithm (log) laws. These laws closely parallel each other.

Exponent Laws

Let $b > 0$ be our base. Then for any real numbers $m, n$, we have

1. $b^m \cdot b^n = b^{m+n}$
2. $\frac{1}{b^n} = b^{-n}$
3. $\frac{b^m}{b^n} = b^{m-n}$
4. $(b^m)^n = b^{mn}$
5. $\sqrt[n]{b} = b^{1/n}$, when $n \neq 0$
6. $\left(\sqrt[n]{b}\right)^m = b^{m/n}$, when $n \neq 0$.

Equation (1) comes from writing out what $b^m$ and $b^n$ mean:

\[
b^m \cdot b^n = \underbrace{b \cdot b \cdots b}_{m \text{ times}} \cdot \underbrace{b \cdot b \cdots b}_{n \text{ times}} = \underbrace{b \cdot b \cdots b}_{m+n \text{ times}} = b^{m+n}.
\]

Putting (1) and (2) together yields (3), and combining (4) and (5) together gets (6). Let’s see an example of each one:

\[
2^2 \cdot 2^6 = 2^{2+6} = 2^8 = 256, \quad \text{by (1)}
\]
\[
3^{-3} = \frac{1}{3^3} = \frac{1}{27}, \quad \text{by (2)}
\]
\[
\frac{5^{10}}{5^7} = 5^{10-7} = 5^3 = 125, \quad \text{by (3)}
\]
\[
(8^3)^3 = (2^3)^3 = 2^{3 \cdot 3} = 2^9 = 512, \quad \text{by (4)}
\]
\[
8^{1/3} = \sqrt[3]{8} = 2, \quad \text{by (5)}
\]
\[
\sqrt[3]{4,096} = \sqrt[3]{2^{12}} = 2^{12/3} = 2^4 = 16, \quad \text{by (6)}.
\]

For completeness, we also list a few more properties about exponents (let $a > 0$ be another base):

\[
(a \cdot b)^m = a^m \cdot b^m \quad \text{(7)}
\]
\[
\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad \text{(8)}
\]
\[
b^0 = 1 \quad \text{(9)}
\]
Logarithm Laws

As before, let $b > 0, b \neq 1$ be our base. Then for any $x, y > 0$, and for any $n$, we have

$$\log_b (x \cdot y) = \log_b x + \log_b y \quad (10)$$

$$\log_b \left( \frac{1}{y} \right) = -\log_b y \quad (11)$$

$$\log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y \quad (12)$$

$$\log_b (x^n) = n \log_b x \quad (13)$$

$$\log_b \left( \sqrt[n]{x} \right) = \frac{1}{n} \cdot \log_b x \quad (14)$$

Equation (10) comes from (1), (11) comes from (2), and so on; we have presented these laws in an order that corresponds with the exponential laws as before. The logarithms themselves are analogous to the exponents in the previous section. This is in line with the fact that the $\log_b$ function “retrieves” the exponent, $x$, from $b^x$, i.e. $\log_b (b^x) = x \log_b b = x$. Let’s give two examples that demonstrate these laws in action:

$$\log_6 81 + \log_6 16 = \log_6 (9^2) + \log_6 (4^2)$$
$$= 2 \log_6 9 + 2 \log_6 4 \quad \text{by (13)}$$
$$= 2 (\log_6 9 + \log_6 4)$$
$$= 2 \log_6 36 \quad \text{by (10)}$$
$$= 2(2) = 4;$$

$$\log_8 2 + 2 \log_8 16 = \log_8 2 + \log_8 256 \quad \text{by (13)}$$
$$= \log_8 512 \quad \text{by (10)}$$
$$= 3.$$

In the second example, we could not immediately combine $\log_8 2$ and $2 \log_8 16$ by (10) because the coefficients in front of the two logarithms are different. To use (10), the logarithms must have the same bases and the same coefficients.