Exponent and Log Laws

Introduction

In this handout, we'll discuss exponent laws and logarithm (log) laws. These laws closely parallel each other.

Exponent Laws

Let $b > 0$ be our base. Then for any real numbers m, n, we have

$$b^m \cdot b^n = b^{m+n} \tag{1}$$

$$\frac{1}{b^n} = b^{-n} \tag{2}$$

$$\frac{b^m}{b^n} = b^m \left(\frac{1}{b^n}\right) = b^{m-n} \tag{3}$$

$$(b^m)^n = b^{m \cdot n} \tag{4}$$

$$\sqrt[n]{b} = b^{1/n}\,,\ \text{when } n \neq 0 \tag{5}$$

$$\sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m = b^{m/n}\,,\ \text{when } n \neq 0. \tag{6}$$

Equation (1) comes from writing out what b^m and b^n mean:

$$b^m \cdot b^n = \overbrace{(b \cdot b \cdots b)}^{m \text{ times}} \cdot \overbrace{(b \cdot b \cdots b)}^{n \text{ times}} = \overbrace{b \cdot b \cdots b}^{m+n \text{ times}} = b^{m+n}\,.$$

Putting (1) and (2) together yields (3), and combining (4) and (5) together gets (6). Let's see an example of each one:

$$\begin{aligned}
2^2 \cdot 2^6 &= 2^{2+6} = 2^8 = 256\,, && \text{by (1)}\\
3^{-3} &= \frac{1}{3^3} = \frac{1}{27}\,, && \text{by (2)}\\
\frac{5^{10}}{5^7} &= 5^{10-7} = 5^3 = 125\,, && \text{by (3)}\\
\left(8^3\right) &= \left(2^3\right)^3 = 2^{3\cdot 3} = 2^9 = 512\,, && \text{by (4)}\\
8^{1/3} &= \sqrt[3]{8} = 2\,, && \text{by (5)}\\
\sqrt[3]{4,096} &= \sqrt[3]{2^{12}} = 2^{12/3} = 2^4 = 16\,, && \text{by (6).}
\end{aligned}$$

For completeness, we also list a few more properties about exponents (let $a > 0$ be another base):

$$(a \cdot b)^m = a^m \cdot b^m \tag{7}$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \tag{8}$$

$$b^0 = 1\,. \tag{9}$$

Logarithm Laws

As before, let $b > 0, b \neq 1$ be our base. Then for any $x, y > 0$, and for any n, we have

$$\log_b(x \cdot y) = \log_b x + \log_b y \tag{10}$$

$$\log_b\left(\frac{1}{y}\right) = -\log_b y \tag{11}$$

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y \tag{12}$$

$$\log_b\left(x^n\right) = n \log_b x \tag{13}$$

$$\log_b\left(\sqrt[n]{x}\right) = \frac{1}{n} \cdot \log_b x\,. \tag{14}$$

Equation (10) comes from (1), (11) comes from (2), and so on; we have presented these laws in an order that corresponds with the exponential laws as before. The *logarithms themselves* are analogous to the exponents in the previous section. This is in line with the fact that the $\log_b$ function "retrieves" the exponent, x, from b^x, i.e. $\log_b\left(b^x\right) = x \log_b b = x$. Let's give two examples that demonstrate these laws in action:

$$\begin{aligned}
\log_6 81 + \log_6 16 &= \log_6\left(9^2\right) + \log_6\left(4^2\right) \\
&= 2\log_6 9 + 2\log_6 4 && \text{by (13)} \\
&= 2\left(\log_6 9 + \log_6 4\right) \\
&= 2\log_6 36 && \text{by (10)} \\
&= 2(2) = 4\,;
\end{aligned}$$

$$\begin{aligned}
\log_8 2 + 2\log_8 16 &= \log_8 2 + \log_8 256 && \text{by (13)} \\
&= \log_8 512 && \text{by (10)} \\
&= 3\,.
\end{aligned}$$

In the second example, we could not immediately combine $\log_8 2$ and $2\log_8 16$ by (10) because the coefficients in front of the two logarithms are different. To use (10), the logarithms must have the same bases and the same coefficients.