Exponential and Logarithmic Functions

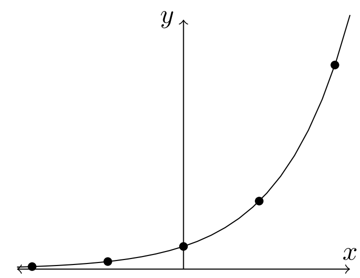
Exponential Functions

Before we talk about logarithmic functions, or logarithms, we must review exponential functions. An **exponential function** is a function of the form

$$f(x) = b^x \,,$$

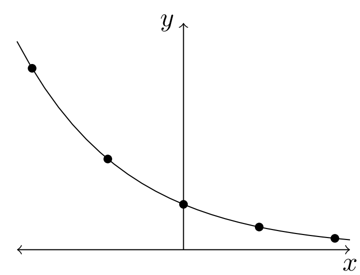
where b is any positive real number except 1. We call b the **base** and x the **exponent**. We will graph two exponential functions. First, let's graph $f(x) = 3^x$ by plotting points:

x	$f(x) = 3^x$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9



Next, let's graph $g(x) = \left(\frac{1}{2}\right)^x$ by plotting points:

x	$g(x) = \left(\frac{1}{2}\right)^x$
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$



Notice the following:

1. Both $f(x)$ and $g(x)$ are never negative or zero.
2. For both $f(x)$ and $g(x)$, the y-intercept is at $(0, 1)$.
3. As the x-values get larger, $f(x)$ gets very large, but $g(x)$ approaches 0.
4. As the x-values get more negative, $f(x)$ approaches 0, but $g(x)$ gets very large.

The graph $f(x) = 3^x$ slopes upward because the base is greater than 1, so multiplying the base by itself results in a larger number. The graph of $g(x) = \left(\frac{1}{2}\right)^x$ on the other hand slopes downward because the base is less than 1, so multiplying the base by itself yields a smaller number. In general, if the base is greater than 1, the graph will slope upwards, and if the base is less than 1, the graph will slope downwards. Note that if the base equals 1, we just get a flat graph since $1^x = 1$ for all real values of x. This is why we excluded it when we defined exponential functions.

Logarithmic Functions

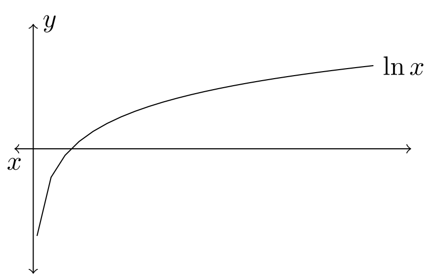
A logarithmic function is the inverse of an exponential function. What does this mean? Suppose $x = b^y$. Here, y is the exponent, and b is the base ($b \neq 1$, and as before we must have $b > 0$). Now we can define the **logarithm**:

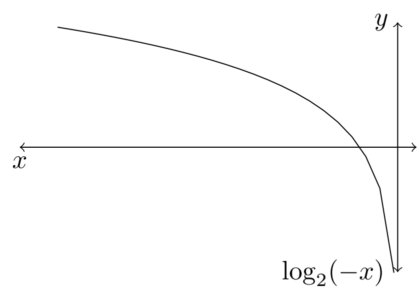
$$x = b^y \text{ is equivalent to } \log_b x = y, \text{ read "log base } b \text{ of } x\text{".}$$

For the logarithm, b is again the **base**, and x is the **argument**. The logarithmic function $\log_b x$ is the inverse of the exponential function in the sense that it takes in the answer you get from exponentiating (here x in $x = b^y$) and retrieves the exponent, y. This may be more easily demonstrated through examples:

$$\begin{aligned}
8 = 2^3 &\longleftrightarrow 3 = \log_2 8 \\
9 = 3^2 &\longleftrightarrow 2 = \log_3 9 \\
1,000 = 10^3 &\longleftrightarrow 3 = \log_{10} 1,000 \\
1 = 3^0 &\longleftrightarrow 0 = \log_3 1 \\
b = b^1 &\longleftrightarrow 1 = \log_b b\,.
\end{aligned}$$

For any $b > 0$ with $b \neq 1$, $\log_b 1 = 0$ because $b^0 = 1$ for positive b. This fact for logarithms parallels the fact for exponential functions that the y-intercept is always at $(0, 1)$. The logarithm (for any base) of 0 or of a negative number does not exist, which parallels the fact that b^x is positive for all $b > 0$. Below we plot two logarithm functions, $f(x) = \log_e x$ (also denoted $\ln x$) and $g(x) = \log_2(-x)$:





Recall that $e \approx 2.718$. The plot on the left, $f(x) = \ln x$, is steeper around $x = 0$ and is always increasing, but with a rate of growth that falls off as $x \to \infty$. As previously mentioned, logairthms are not defined for 0 or for negative numbers. The plot on the right is defined for negative values of x because the argument, $-x$, is positive only when x is negative. When computing the logarithms, it's very helpful to remember the equivalence with exponential functions discussed above.