Factorial Notation

What is Factorial Notation?

Many mathematical applications, especially probability and statistics, require us to multiply consecutive numbers, for example with expressions such as $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. Factorial notation gives us a more convenient way to express this product. For any nonnegative integer n, we denote "n factorial" by $n!$. For example, we have:

$$\begin{aligned} 1! &= 1 \\ 2! &= 2 \cdot 1 = 2 \\ 3! &= 3 \cdot 2 \cdot 1 = 6 \\ 4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24\,. \end{aligned}$$

0 is a special case, which we just define as $0! = 1$. There is not a deep mathematical reason behind this–it is just convenient in applications for us to define $0!$ this way. For a general positive integer n, we have

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1\,.$$

Factorials grow very quickly: $5! = 120$, whereas $10! = 3,628,800$.

Simplifying Factorials

Fractions with factorials, such as the expression $\frac{10!}{8!}$, are especially common. Since 8 and 10 are relatively small numbers, we can write out

$$\frac{10!}{8!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{3,628,800}{40,320} = 90\,.$$

We can cancel out terms that are both in the numerator and the denominator:

$$\frac{10!}{8!} = \frac{10 \cdot 9 \cdot \cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 10 \cdot 9 = 90\,.$$

Writing this out by hand quickly becomes tedious, however. Thankfully, we can instead observe that $10! = 10 \cdot 9 \cdot 8!$ and get

$$\frac{10!}{8!} = \frac{10 \cdot 9 \cdot 8!}{8!} = \frac{10 \cdot 9 \cdot \cancel{8!}}{\cancel{8!}} = 10 \cdot 9 = 90\,.$$

Here are a few more examples:

$$\begin{aligned} \frac{16!}{13!} &= \frac{16 \cdot 15 \cdot 14 \cdot \cancel{13!}}{\cancel{13!}} = 16 \cdot 15 \cdot 14 = 3,360 \\ \frac{29!}{31!} &= \frac{\cancel{29!}}{31 \cdot 30 \cdot \cancel{29!}} = \frac{1}{31 \cdot 30} = \frac{1}{930} \\ \frac{10!}{3! \cdot 7!} &= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{3! \cdot \cancel{7!}} = \frac{10 \cdot 9 \cdot 8}{3!} = \frac{720}{6} = 120\,. \end{aligned}$$

In each case, we expand the largest factorial until we get to the smaller factorial: we expand 16! until we get to 13! and 31! until we get to 29!. In the third one, we expand the largest factorial until we get the second largest factorial.

We may use this method for simplifying general expressions with $n!$ as well. Consider, for example, the expression $\frac{n!}{(n-2)!}$. Recalling our definition of $n!$, we get

$$n! = n(n-1)(n-2)(n-3)\cdots 3 \cdot 2 \cdot 1 = n(n-1)(n-2)!\,.$$

Applying this to our fraction gives

$$\frac{n!}{(n-2)!} = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = n(n-1)\,.$$

We end with a few more examples:

$$\begin{aligned}
\frac{(n+2)!}{n!} &= \frac{(n+2)(n+1)\cancel{n!}}{\cancel{n!}} &&= (n+2)(n+1) \\
\frac{(3n-2)!}{(3n+1)!} &= \frac{\cancel{(3n-2)!}}{(3n+1)(3n)(3n-1)\cancel{(3n-2)!}} &&= \frac{1}{(3n+1)(3n)(3n-1)} \\
\frac{(n+1)!(2n)!}{(n-1)!(2n+2)!} &= \frac{\big((n+1)n\cancel{(n-1)!}\big)\cancel{(2n)!}}{\cancel{(n-1)!}\big((2n+2)(2n+1)\cancel{(2n)!}\big)} &&= \frac{(n+1)n}{(2n+2)(2n+1)}\,.
\end{aligned}$$