Second-Order Equations: Constant Coefficients

Summary

For differential equations of the form $ay'' + by' + cy = 0$, where a, b, and c are constants, we start with the guess $y = e^{rt}$. There are three cases:

Type	Discriminant	General solution
Distinct roots	$\sqrt{b^2 - 4ac} > 0$	$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
Equal roots	$\sqrt{b^2 - 4ac} = 0$	$y(t) = c_1 e^{rt} + c_2 t e^{rt}$
Imaginary roots	$\sqrt{b^2 - 4ac} < 0$	$y(t) = e^{\alpha t} \left(c_1 \cos(\beta t) + c_2 \sin(\beta t) \right)$

In the third case, Imaginary roots, we use $r = \alpha \pm \beta i$.

Identification

As the title suggests, use this method for linear second-order homogeneous equations with constant coefficients. These are equations of the form

$$ay'' + by' + cy = 0\,,$$

where a, b, and c are constants (so no t's here). For equations of the above form, we assume that our solution takes the form $y = e^{rt}$, where r is a constant for which we must solve (in fact, there are often two r values). Calculating y' and y'' with this assumption and then plugging into the above equation gives

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0\,,$$

and dividing both sides by e^{rt} (which is never 0) gives

$$ar^2 + br + c = 0\,.$$

Recall from algebra that this equation has at most two solutions, with

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,,$$

which is the familiar quadratic formula. From here, there are three possibilities: distinct roots, equal roots, and imaginary roots.

Distinct Roots: $\mathbf{b^2 - 4ac > 0}$

Consider the equation $y'' + 4y' - 5 = 0$. Using the guess $y = e^{rt}$ and following the reasoning above gives us

$$r^2 + 4r - 5 = 0\,,$$

which we can solve using the quadratic formula or by factoring the above equation into $(r + 5)(r - 1)$ to get that $r = -5, 1$. We then have two solutions for y:

$$y = e^{-5t} \quad \text{and} \quad y = e^{t}\,.$$

These are two linearly independent solutions, so our general solution is

$$y(t) = c_1 e^{-5t} + c_2 e^{t},$$

where c_1 and c_2 are arbitrary constants.

Equal Roots: $\mathbf{b^2 - 4ac = 0}$

Now consider the equation $y'' - 2y' + y = 0$. This gives us

$$r^2 - 2r + 1 = (r-1)^2 = 0\,,$$

so $r = 1$. Thus, $y = e^t$ is one solution. However, we need a second, linearly independent solution. Using the method of reduction of order (in the handout of the same name), we use this first solution to determine that $y(t) = te^t$ is another solution to the above equation. Our general solution then takes the form

$$y(t) = c_1 e^t + c_2 t e^t\,.$$

Imaginary Roots: $\mathbf{b^2 - 4ac < 0}$

Let's now consider the equation $y'' + y = 0$. This gives $r^2 + 1 = 0$. Plugging this into the quadratic formula, with $a = c = 1$ and $b = 0$, gives

$$r = \frac{\pm\sqrt{-4}}{2} = \frac{\pm 2i}{2} = \pm i\,.$$

This gives us the two solutions $y = e^{it}$ and $y = e^{-it}$, which gives us

$$y(t) = Ae^{it} + Be^{-it}\,,$$

where A and B are arbitrary (complex) constants. In order to find real solutions, we must first use Euler's formula:

$$e^{\lambda \pm i\mu} = e^{\lambda t}\left(\cos(\mu t) \pm i \sin(\mu t)\right)\,,$$

where λ and μ are real numbers. Here, $\lambda = 0$ and $\mu = 1$, so we get

$$\begin{aligned} y(t) &= A\left(\cos t + i \sin t\right) + B\left(\cos t - i \sin t\right) \\ &= (A+B)\cos t + (A-B) i \sin t\,. \end{aligned}$$

Since A and B are arbitrary complex constants, we can actually make the quantities $(A+B)$ and $(A-B)i$ whatever we like, so we may as well replace them with c_1 and c_2:

$$y(t) = c_1 \cos t + c_2 \sin t\,.$$

DISCLAIMER: This handout uses notation and methods from the textbook commonly used for M 427J courses taught at the University of Austin: Braun, Martin, *Differential Equations and Their Applications*, 4th ed. Springer

December 5, 1992.