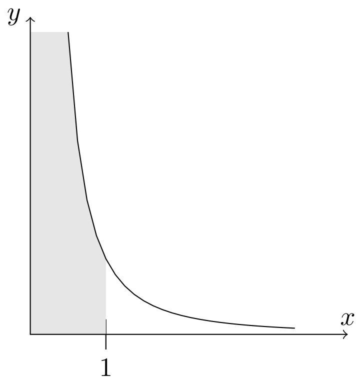
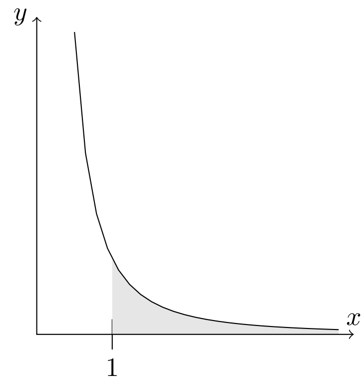
Improper Integrals

Integrals with ∞

Consider the function $f(x) = 1/x^2$. This function goes to 0 as $x \to \pm\infty$ and has a vertical asymptote at $x = 0$. We can take definite integrals such as $\int_1^3 f(x)\,\mathrm{d}x$ and $\int_{-100}^{-2} f(x)\,\mathrm{d}x$ using the Fundamental Theorem of Calculus. What if we wanted to find the area under the curve between the y-axis and $x = 1$ or from $x = 1$ to ∞ (pictured below)?





In other words, suppose we wanted to evaluate

$$\int_0^1 \frac{1}{x^2}\,\mathrm{d}x\,, \quad \text{and} \quad \int_1^\infty \frac{1}{x^2}\,\mathrm{d}x\,.$$

The left-hand integral represents the area in the figure on the left, and the right-hand integral represents the area in the figure on the right. In both figures, the curve will eventually approach the y-axis or the x-axis respectively, so both regions will taper off. We'll see that these two integrals are very different. Let's start with the first integral, then we'll do the second.

The First Integral

We can do this integral by evaluating the limit

$$\lim_{s\to 0^+} \int_s^1 \frac{1}{x^2}\,\mathrm{d}x\,.$$

We now evaluate the integral as normal, while keeping in mind that we have to evaluate the limit at the end:

$$\begin{aligned}
\lim_{s\to 0^+} \int_s^1 \frac{1}{x^2}\,\mathrm{d}x &= \lim_{s\to 0^+} \left(-\frac{1}{x}\bigg|_s^1 \right) \\
&= \lim_{s\to 0^+} \left(-\frac{1}{1} + \frac{1}{s} \right) \\
&= \lim_{s\to 0^+} \left(\frac{1}{s} - 1 \right) = \infty\,.
\end{aligned}$$

We see that this limit is ∞. This indicates that the area in the figure on the left (i.e. the area under the curve $1/x^2$ from 0 to 1) is infinite.

The Second Integral

As before, we evaluate a limit, though this time our limit takes the place of ∞:

$$\lim_{s\to\infty} \int_1^s \frac{1}{x^2}\,\mathrm{d}x\,.$$

Again, we evaluate the integral as normal and evaluate the limit afterwards:

$$\begin{aligned}\lim_{s\to\infty}\int_1^s \frac{1}{x^2}\,\mathrm{d}x &= \lim_{s\to\infty}\left(-\frac{1}{x}\bigg|_1^s\right)\\ &= \lim_{s\to\infty}\left(-\frac{1}{s}+\frac{1}{1}\right)\\ &= \lim_{s\to\infty}\left(1-\frac{1}{s}\right) = 1\,.\end{aligned}$$

This tells us that the area in the right-hand figure (i.e. the area under the curve $1/x^2$ from 1 to ∞) is finite! In fact, it is 1.

Identifying Improper Integrals

There are two main cases for improper integrals:

1. At least one of the limits of integration is $\pm\infty$.
2. The integrand (the function $f(x)$ being integrated) is undefined at or between the bounds of integration.

In the example above, the left integral was of the second type, where the integrand was not defined at the lower bound $x = 0$. The right integral was of the first type, since the upper bound was $+\infty$. A given integral might be improper in both these ways. Let's see two more examples of improper integrals.

Example 2. Evaluate the integral $\int_{-1}^{1} 1/\sqrt[3]{x}\,\mathrm{d}x$.

The integrand is not defined at $x = 0$. 0 falls in between our bounds of integration. To handle this, we split our integral into two parts and evaluate each separately:

$$\int_{-1}^{1}\frac{1}{\sqrt[3]{x}}\,\mathrm{d}x = \int_{-1}^{0} x^{-\frac{1}{3}}\,\mathrm{d}x + \int_{0}^{1} x^{-\frac{1}{3}}\,\mathrm{d}x\,.$$

We can handle both of these integrals using the method described above:

$$\begin{aligned}\int_{-1}^{0} x^{-\frac{1}{3}}\,\mathrm{d}x + \int_{0}^{1} x^{-\frac{1}{3}}\,\mathrm{d}x &= \lim_{s\to 0^-}\int x^{-\frac{1}{3}}\,\mathrm{d}x + \lim_{s\to 0^+}\int x^{-\frac{1}{3}}\,\mathrm{d}x\\ &= \lim_{s\to 0^-}\left(\frac{3}{2}x^{\frac{2}{3}}\bigg|_{-1}^{s}\right) + \lim_{s\to 0^+}\left(\frac{3}{2}x^{\frac{2}{3}}\bigg|_{s}^{1}\right)\\ &= \lim_{s\to 0^-}\left(\frac{3}{2}\left(s^{\frac{2}{3}}-1\right)\right) + \lim_{s\to 0^+}\left(\frac{3}{2}\left(1-s^{\frac{2}{3}}\right)\right)\\ &= \frac{3}{2}(0-1)+\frac{3}{2}(1-0) = 0\,.\end{aligned}$$

Example 3. Evaluate the integral $\int_{-1}^{1} 1/\left(1-x^2\right)\,\mathrm{d}x$.

The integrand is defined at neither of the bounds of integration, but it is defined everywhere else. We thus use an individual limit for each bound:

$$\begin{aligned}\lim_{s\to -1}\left(\lim_{t\to 1}\int_s^t \frac{1}{1-x^2}\,\mathrm{d}x\right) &= \lim_{s\to -1}\left(\lim_{t\to 1}\left(\sin^{-1}x\bigg|_s^t\right)\right)\\ &= \lim_{s\to -1}\left(\lim_{t\to 1}\left(\sin^{-1}t-\sin^{-1}s\right)\right)\\ &= \lim_{s\to -1}\left(\sin^{-1}(1)-\sin^{-1}s\right)\\ &= \sin^{-1}(1)-\sin^{-1}(-1)\\ &= \frac{\pi}{2}-\left(-\frac{\pi}{2}\right) = \pi\,.\end{aligned}$$

Note that we could have interchanged the limits; that is, we could have also evaluated

$$\lim_{t\to 1}\left(\lim_{s\to -1}\int_s^t \frac{1}{1-x^2}\,\mathrm{d}x\right).$$

Following a similar procedure, we would have

$$\begin{aligned}
\lim_{t\to 1}\left(\lim_{s\to -1}\int_s^t \frac{1}{1-x^2}\,\mathrm{d}x\right) &= \lim_{t\to 1}\left(\lim_{s\to -1}\left(\sin^{-1} x\Big|_s^t\right)\right)\\
&= \lim_{t\to 1}\left(\lim_{s\to -1}\left(\sin^{-1} t - \sin^{-1} s\right)\right)\\
&= \lim_{t\to 1}\left(\sin^{-1} t - \sin^{-1}(-1)\right)\\
&= \sin^{-1}(1) - \sin^{-1}(-1)\\
&= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi\,.
\end{aligned}$$