Infinite Series Tests

Summary

In this handout, we describe several tests for determining whether an infinite series is convergent or divergent. There are still some series whose convergence cannot be determined by these tests, but they allow us to find the convergence of many different types of infinite series. For any particular series, you will need to determine which test to use. We summarize the tests in the following table. Then, we will offer some pointers on when to use each test. Unless otherwise noted, we are working with the infinite series $\sum a_n$.

Test	**How to use**
Divergence Test	If $\lim_{n\to\infty} a_n \neq 0$, then the series **diverges**.
Geometric Series Test	Geometric series are of the form $\sum ar^n$. If $\lvert r\rvert < 1$, the series **converges**. Otherwise, it **diverges**.
Integral Test	Let $f(n) = a_n$. If $\int_1^\infty f(x)$ **converges**, then so does the series, and if the integral **diverges**, then so does the series.
p-Series Test	p-series are of the form $\sum \frac{1}{n^p}$. If $p > 1$, the series **converges**. Otherwise ($p \leq 1$), the series **diverges**.
Simple Comparison Test	Let $\sum a_n$, $\sum b_n$ be series with $a_n \leq b_n$. If $\sum b_n$ **converges**, then so does $\sum a_n$. If $\sum a_n$ **diverges**, then so does $\sum b_n$.
Limit Comparison Test	Let $\sum a_n$, $\sum b_n$ be series. If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is positive and finite, then both series **converge** or both series **diverge**.
Alternating Series Test	If $\sum a_n$ is an alternating series, $a_n > a_{n+1}$, and $\lim_{n\to\infty} a_n = 0$, then the series **converges**.
Root Test	Take $L = \lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert}$. If $L < 1$, the series **converges**. If $L > 1$, the series **diverges**. Otherwise, we can't tell.
Ratio Test	Take $L = \lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$. If $L < 1$, the series **converges**. If $L > 1$, the series **diverges**. Otherwise, we can't tell.

- Since it is relatively simple to take the limit of a_n as $n \to \infty$, always use the Divergence Test. It may be inconclusive, but if the limit of a_n is nonzero, then we quickly see that the series diverges.
- The Integral Test is useful for series with $\ln n$. For example, this test shows that the series $\sum \frac{1}{n \ln n}$ diverges.
- When using the Comparison Tests, the exponents in the numerator and denominator can help indicate which comparison series to use. For example, for the series series $\sum \frac{1}{n^2-2}$, the degree of the polynomial in the denominator suggests that we should compare the series to $\sum \frac{1}{n^2}$, which converges by the p-Series Test. This also works if the exponents differ by 2, for example in $\sum \frac{n^2}{n^4-n^3+1}$.
- The Limit Comparison Test is generally more useful than the Simple Comparison Test.
- For the Alternating Series Test, remember that $\cos(n\pi)$ and $\sin\left(n\pi + \frac{\pi}{2}\right)$ are both equal to $(-1)^n$, so series with such terms are also probably alternating.
- The Root Test is useful when many of the parts of the series are raised to the n^{th} power, such as in $\sum \frac{2^n}{n^n}$. Remember that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

- The Ratio Test is useful when the series has factorials, for example $\sum \frac{2^n n!}{n^n}$. It is also useful when the terms of the series have exponentials such as 3^n or 2^{2n+1}.

Convergence and Divergence of Infinite Series

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series. Then the $\mathbf{N^{th}}$ **partial sum** of this series is defined to be the sum of the first N terms of this series: $\sum_{n=1}^{N} a_n$. Consider the series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. The 3^{rd} partial sum of this series is given by

$$\sum_{n=1}^{3} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}\,.$$

The first partial sum, the second partial sum, etc. of a series form a sequence. For example, the sequence of partial sums for the above series is

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots\,.$$

We can then ask whether this sequence converges or not. There are various formulas for the N^{th} partial sum of the above series. From this sequence, we can see that one such formula is $\frac{2n-1}{2n}$. As $n \to \infty$, this quantity approaches 1. Thus, we say that the series **converges** to 1. In general, a series **converges** if its sequence of partial sums converges. On the other hand, a series **diverges** if its sequence of partial sums diverges.

The fact that the above series converges to 1 means that, as we add on more and more of the terms of the series, we get a number approaching 1. We're being careful with how we say this, since it doesn't make much sense to "add up" the infinitely many terms in the series. Nevertheless, we could write

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1\,.$$

When we defined the N^{th} partial sum of a series, the starting index of our series was 1. This is not necessary, as we can actually start at any index we like, as long as we bear in mind that the N^{th} partial sum might no longer be the sum of the first N terms. For example, the 3^{rd} partial sum of $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is $\sum_{n=0}^{3} \left(\frac{1}{2}\right)^n$, which actually has 4 terms. Nevertheless, no matter where we *start* our index, this will not affect the convergence or divergence of the series.

Testing the Convergence of an Infinite Series

For any given infinite series, it may be tedious or even impossible to get a formula for the sequence of partial sums, so we cannot rely on just writing out the sequence of partial sums and finding the limit. Thankfully, there are many tests for determining the convergence and divergence of an infinite series. We will list them now. For convenience, we will often write $\sum a_n$ instead of $\sum_{n=c}^{\infty} a_n$. Again, the starting index of the series does not affect its convergence or divergence.

Divergence Test. An infinite series $\sum a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$.

Consider the series $\sum 1$. $a_n = 1$ for all n, so $\lim_{n\to\infty} a_n = 1 \neq 0$. Thus, this series diverges. In other words, $1+1+1+\cdots$ does not converge to a fixed number. **IMPORTANT:** the terms of a series approaching 0 does **NOT** mean that the series converges. If the terms of a series approach 0, then we can say nothing about the convergence or divergence of the series yet. For example, the series $\sum \frac{1}{n}$ actually diverges, even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

Geometric Series Test. A **geometric series** is of the form $\sum ar^n$, where a is called the **initial term**, and r is called the **constant ratio**. A geometric series converges if $|r| < 1$, i.e. if $-1 < r < 1$. If, on the other hand, $|r| \geq 1$, then the series diverges. Furthermore, when the series converges, it converges to $\frac{a}{1-r}$.

The series $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$, with $a = 1$ and $r = \frac{1}{2}$, converges by the Geometric Series Test since $\left|\frac{1}{2}\right| < 1$. It has sum $\frac{1}{1-1/2} = 2$. On the other hand, the series $\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n$ diverges, since $|r| = \left|\frac{3}{2}\right| \geq 1$. Also, the alternating series

$$\sum_{n=0}^{\infty}(-1)^n = 1 - 1 + 1 - 1 + \cdots$$

diverges since $|r| = |-1| = 1$. More on alternating series later.

Integral Test. Let $f(x)$ be a positive, continuous, and decreasing function for $x \geq 1$. If $a_n = f(n)$, then (1) if $\int_1^{\infty} f(x)\,\mathrm{d}x$ converges, then so does $\sum_{n=1}^{\infty} a_n$, and (2) if $\int_1^{\infty} f(x)\,\mathrm{d}x$ diverges, then so does $\sum_{n=1}^{\infty} a_n$.

We can use this Test to show that $\sum \frac{1}{n}$ diverges. First note that $f(x) = \frac{1}{x}$ is a positive, continuous, and decreasing function for $x \geq 1$. Then we compute the improper integral $\int_1^{\infty} \frac{1}{x}\,\mathrm{d}x$:

$$\int_1^{\infty} f(x)\,\mathrm{d}x = \lim_{s\to\infty}\int_1^{s}\frac{1}{x}\,\mathrm{d}x = \lim_{s\to\infty}\ln x\Big|_1^s = \lim_{s\to\infty}\ln s = \infty\,.$$

Since this improper integral diverges, so too does the series $\sum \frac{1}{n}$. On the other hand, the improper integral $\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x$ converges, so the series $\sum \frac{1}{n^2}$ converges as well.

p-Series Test. The series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

This Test actually comes from the Integral Test. According to this Test, $\sum \frac{1}{n}$ diverges since $p = 1$, and $\sum \frac{1}{n^2}$ converges since $p = 2 > 1$. This Test also shows that $\sum \frac{1}{\sqrt{n}}$ diverges since $p = \frac{1}{2} \leq 1$, and $\sum \frac{1}{n^{1.1}}$ converges since $1.1 > 1$.

Simple Comparison Test. Let $\sum a_n$ and $\sum b_n$ be two series with $a_n, b_n \geq 0$ and $a_n \leq b_n$ for all n. Then (1) if $\sum b_n$ converges then so does $\sum a_n$, and (2) if $\sum a_n$ diverges, then so does $\sum b_n$.

Informally, $\sum a_n$ is a smaller series than $\sum b_n$ since all the terms are smaller. Thus, if the larger series, $\sum b_n$, converges, then so too must the smaller series. On the other hand, if the smaller series diverges, then so too must the larger series. With this Test, we can say that $\sum \frac{3}{n}$ diverges since $\sum \frac{1}{n}$ diverges and $\frac{1}{n} \leq \frac{3}{n}$ for all n; that is, $\sum \frac{3}{n}$ is a bigger series than the divergent series $\sum \frac{1}{n}$. We also have that $\sum \frac{1}{n^2+2}$ converges, since $\frac{1}{n^2+2} \leq \frac{1}{n^2}$ for all n, and $\sum \frac{1}{n^2}$ converges.

We have to be careful with this Test as it is actually quite limited. For example, it says nothing about $\sum \frac{1}{n+1}$ since $\frac{1}{n+1} < \frac{1}{n}$. Even though $\sum \frac{1}{n}$ diverges, we cannot say anything about the smaller series $\sum \frac{1}{n+1}$ yet. Similarly, we cannot assess the convergence or divergence of $\sum \frac{1}{n^2-2}$. The next Test will show that these series behave as we might expect: the former diverges and the latter converges.

Limit Comparison Test. Let $\sum a_n$ and $\sum b_n$ be two series, with $a_n \geq 0$ and $b_n > 0$ for all n. Consider the limit

$$L = \lim_{n\to\infty}\frac{a_n}{b_n}\,.$$

If L is positive and finite ($0 < L < \infty$), then either both series converge or both series diverge. Note that we may also define L as

$$L = \lim_{n\to\infty}\frac{b_n}{a_n}\,,$$

assuming $a_n > 0$ for all n, since this will be the reciprocal of the previous quantity. This is because the L is positive and finite if and only if $\frac{1}{L}$ is positive and finite.

This Test is much more useful than the Simple Comparison Test. Consider the two series in the last section for which the Simple Comparison Test was inconclusive: $\sum \frac{1}{n+1}$ and $\sum \frac{1}{n^2-2}$. We compare the first series to $\sum \frac{1}{n}$ (which diverges) and the second to $\frac{1}{n^2}$ (which converges):

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} = 1\,,$$

which is positive and finite. Similarly,

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2-2}} = \lim_{n\to\infty} \frac{n^2-2}{n} = 1\,,$$

which is again positive and finite. Thus, $\sum \frac{1}{n+1}$ acts like $\sum \frac{1}{n}$ (i.e. it diverges), and $\sum \frac{1}{n^2-2}$ acts like $\sum \frac{1}{n^2}$ (i.e. it converges). Again, it does not matter which series we choose to go on top or which we choose to go on the bottom.

The most difficult part of the Limit Comparison Test tends to be figuring out which comparison series to use. This is usually done on a case-by-case basis. For example, suppose we want to figure out whether $\sum \frac{2n^2-n+1}{n^4-3n^3-2}$ converges. Looking at the highest-power terms in both the numerator and the denominator, we see that the denominator's highest power is 2 greater than the highest power in the numerator. This suggests comparing the series to $\sum \frac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{\frac{2n^2-n+1}{n^4-3n^3-2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2n^4-n^3+n^2}{n^4-3n^3-2} = 2\,.$$

Since $0 < 2 < \infty$ and $\sum \frac{1}{n^2}$ converges, we conclude that $\sum \frac{2n^2-n+1}{n^4-3n^3-2}$ converges as well. Note that if we had swapped which series went on top and which one went on the bottom in the above limit, we would have gotten a limit of $\frac{1}{2}$, which is again positive and finite. Thus, we would've come to the same concusion.

Alternating Series Test. Let $a_n > 0$ for all n. An **alternating series** is a series of the form $\sum_{n=0}^{\infty}(-1)^n a_n = a_0 - a_1 + a_2 - a_3 + \cdots$, i.e. one in which the terms "alternate" between positive and negative. Now consider a_n. If (1) $a_{n+1} < a_n$ for all n (i.e. if the terms are decreasing), and if (2) $\lim_{n\to\infty} a_n = 0$, then the series $\sum(-1)^n a_n$ converges.

Note that it is permissible for, say, the first 4 terms of the sequence $\{a_n\}$ to not be decreasing so long as the sequence is *eventually* decreasing (for example, after the 5$^{\text{th}}$ term). This particular detail is rarely very important for most problems in M 408D/L, but it is necessary. We can use this Test to determine whether the series $\sum \frac{(-1)^n}{n}$ converges. The $(-1)^n$ makes this an alternating series, with $a_n = \frac{1}{n}$. Furthermore, $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$, so the terms a_n are decreasing for all n. Lastly, $\lim_{n\to\infty} \frac{1}{n} = 0$. We conclude that this series converges by the Alternating Series Test.

Note that this does *not* mean that $\sum \frac{1}{n}$ converges; in fact, we have already seen that it diverges. When it is the case that $\sum(-1)^n a_n$ converges but $\sum |(-1)^n a_n| = \sum a_n$ diverges, we say that $\sum(-1)^n a_n$ **converges conditionally**. If both series converge, we say that $\sum(-1)^n a_n$ **converges absolutely**. For example, $\sum \frac{(-1)^n}{n}$ converges conditionally whereas $\sum \frac{(-1)^n}{n^2}$ converges absolutely since $\sum \frac{1}{n^2}$ also converges.

The expression $\cos(n\pi)$ equals $(-1)^n$. One can verify this by writing out the first several terms of both sequences. Thus, the series $\sum \frac{\cos(n\pi)}{n}$ is actually a candidate for the Alternating Series Test since it is in fact the series $\sum \frac{(-1)^n}{n}$. Similarly, $\sin\left(n\pi + \frac{\pi}{2}\right) = (-1)^n$.

Root Test. Given the series $\sum a_n$, consider the quantity

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}\,.$$

The Root Test states that...

- if $L < 1$, then the series is absolutely convergent.
- if $L > 1$, then the series is divergent.

- if $L = 1$, then the Test is inconclusive.

Two important facts are: $\lim_{n\to\infty} \sqrt[n]{n} = 1$, and $\lim_{n\to\infty} \sqrt[n]{k} = 1$ for any constant k. We will test the convergence of the series $\sum_{n=1}^{\infty}(-1)^n n^2 \left(\frac{2n+3}{3n+2}\right)^n$. We compute

$$\begin{aligned}
L = \lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\left|(-1)^n n^2 \left(\frac{2n+3}{3n+2}\right)^n\right|} \\
&= \lim_{n\to\infty} \sqrt[n]{n^2 \left(\frac{2n+3}{3n+2}\right)^n} \\
&= \lim_{n\to\infty} \sqrt[n]{n^2}\, \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} \\
&= \left(\lim_{n\to\infty} \left(\sqrt[n]{n}\right)^2\right) \cdot \left(\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n}\right) \\
&= (1)^2 \lim_{n\to\infty} \frac{2n+3}{3n+2} \\
&= \frac{2}{3}\,,
\end{aligned}$$

so this series converges by the Root Test since $L = \frac{2}{3} < 1$. Notice how the n^{th} root cancels out the exponent of n. Series that have quantities raised to the n^{th} power are candidates for the Root Test. Remember that the Test says nothing if $L = 1$. For example, the Root Test says nothing about the series $\sum n$, since $L = 1$ in this case (recall that $\sqrt[n]{n} \to 1$ as $n \to \infty$). We easily see that this series diverges by the Divergence Test, however, since the terms do not go to 0.

Ratio Test. Given the series $\sum a_n$, consider the quantity

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

The Ratio Test states that...

- if $L < 1$, then the series is absolutely convergent.
- if $L > 1$, then the series is divergent.
- if $L = 1$, then the Test is inconclusive.

Notice how the hypotheses are similar to those for the Root Test, though the quantity L is different. Recall that a_{n+1} is obtained by plugging in $n+1$ for n in a_n. So if $a_n = n!$, then $a_{n+1} = (n+1)!$. The Ratio Test is useful for a variety of series, especially for those with factorials in them. Let's first determine the convergence of the series $\sum \frac{n^2}{n!}$. We compute

$$\begin{aligned}
L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left(\frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}}\right) \\
&= \lim_{n\to\infty} \left(\frac{(n+1)^2}{n^2} \cdot \frac{n!}{(n+1)!}\right) \\
&= \left(\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2\right) \cdot \left(\lim_{n\to\infty} \frac{1}{n+1}\right) \\
&= (1)(0) \\
&= 0\,,
\end{aligned}$$

so the series converges by the Ratio Test since $L = 0 < 1$.

For another example, suppose we want to determine the convergence of the series $\sum(-1)^n \frac{n!}{n^n}$. We can't really use the Root Test since it's difficult to evaluate $\sqrt[n]{n!}$. It would be difficult to try and use the Alternating Series Test since it's not quite clear what the terms approach as $n \to \infty$. Let's use the Ratio Test. We compute

$$\begin{aligned}
L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left(\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right) \\
&= \lim_{n\to\infty} \left(\frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}}\right) \\
&= \lim_{n\to\infty} \left((n+1)\frac{n^n}{(n+1)^{n+1}}\right) \\
&= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n .
\end{aligned}$$

This limit is difficult to evaluate. The limit of the reciprocal is

$$\frac{1}{L} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e\,.$$

Thus, $L = e^{-1} < 1$. The series, therefore, converges by the Ratio Test. The last equality follows since the preceding limit is the *definition* of the number e. This limit is also covered in detail in Example 6 of the handout on L'Hôpital's Rule.