Integration by Parts

Summary

The formula for **Integration by Parts** is

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u\,.$$

When integrating by parts, we choose a part of the integrand (the expression being integrated) to be u and the rest of the integrand to be $\mathrm{d}v$. We derive u and integrate $\mathrm{d}v$ to get $\mathrm{d}u$ and v. We then plug in all these parts into our formula.

When choosing a function for u, follow the **ILATE rule**:

$$\begin{aligned}
\text{I} &= \text{Inverse trigonometric} : \sin^{-1} x, \tan^{-1} x \\
\text{L} &= \text{Logarithmic} : \ln x \\
\text{A} &= \text{Algebraic} : 3, x, 2x^2, 4x^3 + 3x \\
\text{T} &= \text{Trigonometric} : \cos x, \sin x \\
\text{E} &= \text{Exponential} : e^x\,.
\end{aligned}$$

In $\int (2x+5)e^x\,\mathrm{d}x$, for example, set $u = 2x+5$, since $2x+5$ is algebraic (A) and so takes priority over e^x, which is exponential (E). $\mathrm{d}v$ becomes $e^x\,\mathrm{d}x$. Integration by parts may be used more than once. Examples 3 and 4 illustrate this. Each iteratiion will likely use different functions for u and $\mathrm{d}v$, so be sure not to get them confused between iterations!

Integration by Parts and the Product Rule

Integration by parts will allow us to solve integrals of the product of two different functions, such as $\int x\cos x\,\mathrm{d}x$. The idea behind integration by parts comes from the product rule for derivatives. Let u and v be functions of x. Then

$$[uv]' = u'v + uv'\,.$$

If we subtract $u'v$ from both sides (and switch around the equality), this gives

$$uv' = [uv]' - u'v\,.$$

Integrating both sides of this equation suggets the formula for **Integration by Parts**, which is as follows:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u\,.$$

Using Integration by Parts

Let's see how we can use this formula to evaluate $\int x\cos x\,\mathrm{d}x$. We write

$$u = x\,, \quad \mathrm{d}v = \cos x\,\mathrm{d}x\,.$$

We need to get $\mathrm{d}u$ and v. To do this, we differentiate u and integrate $\mathrm{d}v$:

$$\mathrm{d}u = \mathrm{d}x\,, \quad v = \sin x\,.$$

Now we plug these four parts into our formula:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$
$$\int x\cos x\,\mathrm{d}x = x\sin x - \int \sin x\,\mathrm{d}x\,.$$

The integral on the right-hand side is easy to integrate: it is $-\cos x$. This gives us a final answer of

$$\int x\cos x\,\mathrm{d}x = x\sin x + \cos x + C\,.$$

We could have just as easily chosen $u = \cos x$ and $\mathrm{d}v = x\,\mathrm{d}x$. This wouldn't have helped us, however, as this would get us

$$\int x\cos x\,\mathrm{d}x = \frac{1}{2}x^2\cos x + \int \frac{1}{2}x^2\sin x\,\mathrm{d}x\,,$$

and the integral on the right-hand side is just as bad as the one on the left! Try to compute $\mathrm{d}u$ and v and plug them into the integration by parts formula to get what we have here. This example shows that what we choose for u and $\mathrm{d}v$ are important when solving the integral.

Choosing u and dv

When using the integration by parts formula, we take the derivative of u and take the integral of $\mathrm{d}v$. As a general rule, we want $v\,\mathrm{d}u$ to be easier to integrate than $u\,\mathrm{d}v$. We can accomplish this by choosing u such that it becomes a simpler function when we take the derivative of it. In the last example, we chose $u = x$ since taking the derivative of it just gives $\mathrm{d}x$. The $\mathrm{d}v$ term becomes whatever is left over from our choice of u. Remember the acronym **ILATE** when choosing a suitable u. ILATE stands for

$$\begin{aligned}
\text{I} &= \text{Inverse trigonometric} : \sin^{-1}x, \tan^{-1}x \\
\text{L} &= \text{Logarithmic} : \ln x \\
\text{A} &= \text{Algebraic} : 3, x, 2x^2, 4x^3+3x \\
\text{T} &= \text{Trigonometric} : \cos x, \sin x \\
\text{E} &= \text{Exponential} : e^x
\end{aligned}$$

Note that we treat constants as algebraic functions. Let's do some examples to illustrate the ILATE rule.

Example 1. $\int x\ln x\,\mathrm{d}x$.

Using ILATE, logarithmic ($\ln x$) comes before algebraic (x), so we have

$$\begin{aligned}
u &= \ln x & \mathrm{d}v &= x\,\mathrm{d}x \\
\mathrm{d}u &= \frac{1}{x}\,\mathrm{d}x & v &= \frac{1}{2}x^2\,,
\end{aligned}$$

and plugging this into the integration by parts formula gives

$$\int x\ln x\,\mathrm{d}x = \frac{1}{2}x^2\ln x - \int\left(\frac{1}{2}x^2\right)\left(\frac{1}{x}\,\mathrm{d}x\right) = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C\,.$$

Example 2. $\int 2\tan^{-1}x\,\mathrm{d}x$.

Using ILATE, inverse trig ($\tan^{-1}x$) takes priority over algebraic (2), giving us

$$\begin{aligned}
u &= \tan^{-1}x & \mathrm{d}v &= 2\,\mathrm{d}x \\
\mathrm{d}u &= \frac{1}{1+x^2}\,\mathrm{d}x & v &= 2x\,,
\end{aligned}$$

which then gives us

$$\int 2\tan^{-1}x\,\mathrm{d}x = 2x\tan^{-1}x - \int\frac{2x}{1+x^2}\,\mathrm{d}x = 2x\tan^{-1}x - \ln\left(1+x^2\right) + C\,.$$

Note that we solve the integral in the second expression using u-substitution with $1+x^2$. Don't confuse the "u" in u-substitution with the u in integration by parts!

Example 3. $\int x^2e^x\,\mathrm{d}x$.

Using ILATE, we see that we must use

$$\begin{aligned} u &= x^2 & \mathrm{d}v &= e^x\,\mathrm{d}x \\ \mathrm{d}u &= 2x\,\mathrm{d}x & v &= e^x\,, \end{aligned}$$

giving us

$$\int x^2 e^x\,\mathrm{d}x = x^2 e^x - \int 2xe^x\,\mathrm{d}x\,.$$

How do we handle the integral on the right-hand side? We simply do integration by parts again. If we just focus on $\int 2xe^x\,\mathrm{d}x$, we get a new u and $\mathrm{d}v$:

$$\begin{aligned} u &= 2x & \mathrm{d}v &= e^x\,\mathrm{d}x \\ \mathrm{d}u &= 2\,\mathrm{d}u & v &= e^x\,, \end{aligned}$$

so that

$$\int 2xe^x\,\mathrm{d}x = 2xe^x - \int 2e^x\,\mathrm{d}x = 2xe^x - 2e^x + C\,.$$

We now get our final answer by plugging this in:

$$\begin{aligned} \int x^2 e^x\,\mathrm{d}x &= x^2 e^x - \int 2xe^x\,\mathrm{d}x \\ &= x^2 e^x - [2xe^x - 2e^x] \\ &= x^2 e^x - 2xe^x + 2e^x + C\,. \end{aligned}$$

Example 4. $\int e^x \cos x\,\mathrm{d}x$.

ILATE tells us that

$$\begin{aligned} u &= \cos x & \mathrm{d}v &= e^x\,\mathrm{d}x \\ \mathrm{d}u &= -\sin x\,\mathrm{d}x & v &= e^x\,, \end{aligned}$$

which gives

$$\int e^x \cos x\,\mathrm{d}x = e^x \cos x + \int e^x \sin x\,\mathrm{d}x\,.$$

For the integral $\int e^x \sin x\,\mathrm{d}x$, we use a new u and $\mathrm{d}v$:

$$\begin{aligned} u &= \sin x & \mathrm{d}v &= e^x\,\mathrm{d}x \\ \mathrm{d}u &= \cos x\,\mathrm{d}x & v &= e^x\,, \end{aligned}$$

giving us

$$\int e^x \sin x\,\mathrm{d}x = e^x \sin x - \int e^x \cos x\,\mathrm{d}x\,.$$

This may not look too promising, but let's plug this back into our original equation:

$$\begin{aligned} \int e^x \cos x\,\mathrm{d}x &= e^x \cos x + \int e^x \sin x\,\mathrm{d}x \\ &= e^x \cos x + \left[e^x \sin x - \int e^x \cos x\,\mathrm{d}x\right] \\ &= e^x \cos x + e^x \sin x - \int e^x \cos x\,\mathrm{d}x\,. \end{aligned}$$

Our desired integral appears on both sides of the equality. If we add $\int e^x \cos x\,\mathrm{d}x$, we get

$$2\int e^x \cos x\,\mathrm{d}x = e^x \cos x + e^x \sin x\,,$$

and dividing both sides by 2 gets us a final answer of

$$\int e^x \cos x\,\mathrm{d}x = \frac{1}{2}e^x \cos x + \frac{1}{2}e^x \sin x\,.$$