The University of Texas at Austin
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Linear, First-Order Equations: Integration Factors

Summary

We use the method of integration factors on linear, first-order differential equations, which are of the form

$$y' + p(t)y = q(t)\,.$$

The method is a 5-step process:

1. Set the coefficient of y' equal to 1 if necessary (by dividing both sides by that coefficient).
2. Multiply both sides of the differential equation by the integrating factor, $\mu = e^{\int p(t)\,\mathrm{d}t}$:
$$e^{\int p(t)\,\mathrm{d}t}y' + p(t)e^{\int p(t)\,\mathrm{d}t}y = e^{\int p(t)\,\mathrm{d}t}q(t)\,.$$
3. Rewrite the left-hand side using the Product Rule for Derivatives:
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\int p(t)\,\mathrm{d}t}y\right] = e^{\int p(t)\,\mathrm{d}t}q(t)\,.$$
4. Integrate both sides, keeping the constant of integration, C:
$$e^{\int p(t)\,\mathrm{d}t}y = \int e^{\int p(t)\,\mathrm{d}t}q(t)\,\mathrm{d}t\,.$$
5. Isolate $y(t)$ to get the final solution.

When to Use Integration Factors

In this handout we'll describe the method of integration factors, which we use to solve linear, first-order differential equations. A differential equation is **first-order** if the highest-power derivative of y is only the first derivative (so no second, third, etc. derivatives). A first-order differential equation is **linear** if it is of the form

$$y' + p(t)y = q(t)\,.$$

This equation only contains the first derivative of y (no higher-order derivatives), so it is first order. The term "linear" comes from the fact that y is only multiplied by some other expression, but beyond that nothing is done to y. Contrast this with another first-order differential equation such as $y' + \cos y = 0$ or $y' + ty^2 = 2$, where the first example takes the cosine of y and the second squares y.

The following method works for *all* linear first-order differential equations and only for such equations. Thus, knowing how to identify a linear first-order differential equation is crucial to knowing when to use the method of integrating factors.

The Method of Integration Factors

Suppose we have our linear first-order differential equation:

$$y' + p(t)y = q(t)\,. \tag{1}$$

We construct our **integration factor**, μ as follows:

$$\mu = e^{\int p(t)\,\mathrm{d}t}\,.$$

Here are some examples of differential equations, with the integration factor (note that the equations are linear and first-order):

$$\begin{aligned}
y' + 3y = 3 &\longrightarrow \mu = e^{\int 3\,\mathrm{d}t} = e^{3t}\\
y' + \frac{1}{t}y = \frac{1}{t^2} &\longrightarrow \mu = e^{\int 1/t\,\mathrm{d}t} = e^{\ln t} = t\\
y' + 2ty = e^{-t^2} &\longrightarrow \mu = e^{\int 2t\,\mathrm{d}t} = e^{t^2}\,.
\end{aligned}$$

Note that when we compute μ, we ignore the constant of integration. There is a deeper mathematical reason for why we can do this, which we omit. What do we do with our integration factor, μ? We multiply both sides of (1) by μ:

$$\begin{aligned} \mu y' + p(t)\mu y &= \mu q(t) \\ e^{\int p(t)\,\mathrm{d}t} y' + p(t) e^{\int p(t)\,\mathrm{d}t} y &= e^{\int p(t)\,\mathrm{d}t} q(t)\,. \end{aligned} \tag{2}$$

Recall that $\mu = e^{\int p(t)\,\mathrm{d}t}$. We defined μ so that the left-hand side of this equation is just the Product Rule for differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mu y\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\int p(t)\,\mathrm{d}t} y\right] = e^{\int p(t)\,\mathrm{d}t} y' + p(t) e^{\int p(t)\,\mathrm{d}t} y\,.$$

We then rewrite the left-hand side of (2) accordingly:

$$e^{\int p(t)\,\mathrm{d}t} y' + p(t) e^{\int p(t)\,\mathrm{d}t} y = [e^{\int p(t)\,\mathrm{d}t} y]'\,.$$

which gives us

$$\left[e^{\int p(t)\,\mathrm{d}t} y\right]' = e^{\int p(t)\,\mathrm{d}t} q(t)\,.$$

From here, we can integrate both sides with respect to t and retrieve y that way. Let's do this for the three previously mentioned examples.

Example 1. Solve $y' + 3y = 3$.

We already computed the integrating factor $\mu = e^{3t}$. We multiply both sides of the differential equation by μ and then integrate:

$$\begin{aligned} y' + 3y &= 3 && \\ e^{3t} y' + 3e^{3t} y &= 3e^{3t} && \text{Multiply both sides by } \mu. \\ \left[e^{3t} y\right]' &= 3e^{3t} && \text{Product Rule for Derivatives.} \\ \int \left[e^{3t} y\right]' \,\mathrm{d}t &= \int 3e^{3t}\,\mathrm{d}t && \text{Integrate both sides.} \\ e^{3t} y &= e^{3t} + C && \\ y &= 1 + Ce^{-3t} && \text{Isolate } y. \end{aligned}$$

Here we *do not neglect* the constant of integration, C. $y = 1 + Ce^{-3t}$ is the general solution to the above differential equation.

Example 2. Solve $y' + \frac{1}{t} y = \frac{1}{t^2}$.

We already computed $\mu = t$. We proceed exactly as before:

$$\begin{aligned} y' + \frac{1}{t} y &= \frac{1}{t^2} && \\ ty' + y &= \frac{1}{t} && \text{Multiply both sides by } \mu. \\ [ty]' &= \frac{1}{t} && \text{Product Rule for Derivatives.} \\ \int [ty]' \,\mathrm{d}t &= \int \frac{1}{t}\,\mathrm{d}t && \text{Integrate both sides.} \\ ty &= \ln t + C && \\ y &= \frac{\ln t}{t} + \frac{C}{t} && \text{Isolate } y. \end{aligned}$$

Example 3. Solve $y' + 2ty = e^{-t^2}$.

We already computed $\mu = e^{t^2}$. We then go through the method:

$$\begin{aligned} y' + 2ty &= e^{-t^2} \\ e^{t^2}y' + 2te^{t^2}y &= 1 && \text{Multiply both sides by } \mu. \\ \left[e^{t^2}y\right]' &= 1 && \text{Product Rule for Derivatives.} \\ \int \left[e^{t^2}y\right]' \mathrm{d}t &= \int 1 \, \mathrm{d}t && \text{Integrate both sides.} \\ e^{t^2}y &= t + C \\ y &= te^{-t^2} + Ce^{-t^2} && \text{Isolate } y. \end{aligned}$$

Example 4. Solve $(\cos t)y' + (\sin t)y = \cos^2 t$, subject to the initial-value condition $y(0) = 6$.

This is a linear first-order equation. However, we must make the coefficient of y' equal to 1 for the method of integration factors to work. Dividing both sides by $\cos t$, we must now solve

$$y' + (\tan t)y = \cos t\,.$$

We compute $\mu = e^{\int \tan t \, \mathrm{d}t} = e^{\ln|\sec t|} = \sec t$. With the integration factor in hand, we follow the usual method:

$$\begin{aligned} y' + (\tan t)y &= \cos t \\ (\sec t)y' + (\sec t \tan t)y &= 1 && \text{Multiply both sides by } \mu. \\ [(\sec t)y]' &= 1 && \text{Product Rule for Derivatives.} \\ \int [(\sec t)y]' \, \mathrm{d}t &= \int 1 \, \mathrm{d}t && \text{Integrate both sides.} \\ (\sec t)y &= t + C \\ y &= t\cos t + C\cos t && \text{Isolate } y. \end{aligned}$$

We now use our initial-value condition to solve for C:

$$\begin{aligned} 6 = y(0) &= (0)\cos(0) + C\cos(0) \\ &= 0 + C(1) \\ &= C\,, \end{aligned}$$

giving us our final answer of

$$y(t) = t\cos t + 6\cos t\,.$$

DISCLAIMER: This handout uses notation and methods from the textbook commonly used for M 427J courses taught at the University of Austin:

Braun, Martin, *Differential Equations and Their Applications*, 4^{th} ed. Springer
December 5, 1992.