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Second-Order Equations: Judicious Guessing

Summary

Use the method of **judicious guessing**, also known as **undetermined coefficients** to find a particular solution $\psi(t)$ to differential equations of the form

$$y' + by' + cy = g(t)\,.$$

Our guess for $\psi(t)$ will depend on the form of $g(t)$, and from our initial guess we compute $\psi'(t)$ and $\psi''(t)$, plug them into the differential equation, and then solve for the coefficients in $\psi(t)$. This table summarizes which $\psi(t)$ to guess according to $g(t)$:

Form of $g(t)$		Guess for $\psi(t)$
$a_0 + \cdots + a_n t^n$	$b \neq 0, c \neq 0$	$A_0 + \cdots + A_n t^n$
	$b \neq 0, c = 0$	$t\,(A_0 + \cdots + A_n t^n)$
	$b = c = 0$	$t^2\,(A_0 + \cdots + A_n t^n)$
$(a_0 + \cdots + a_n t^n)\,e^{\alpha t}$	$e^{\alpha t}$ does not solve the homogeneous equation.	$(A_0 + \cdots + A_n t^n)\,e^{\alpha t}$
	$e^{\alpha t}$ solves the homogeneous equation but not $te^{\alpha t}$.	$t\,(A_0 + \cdots + A_n t^n)\,e^{\alpha t}$
	Both $e^{\alpha t}$ and $te^{\alpha t}$ solve the homogeneous equation.	$t^2\,(A_0 + \cdots + A_n t^n)\,e^{\alpha t}$
$(a_0 + \cdots + a_n t^n)\cos(\beta t)$		$\mathrm{Re}\left((A_0 + \cdots + A_n t^n)\,e^{\beta i t}\right)$
$(a_0 + \cdots + a_n t^n)\sin(\beta t)$		$\mathrm{Im}\left((A_0 + \cdots + A_n t^n)\,e^{\beta i t}\right)$

Guessing Solutions

The method of variation of parameters, while powerful, often requires solving difficult integrals. In some cases, it is better just to make an educated guess for the solution. This method, called **judicious guessing** or **undetermined coefficients**, works when the differential equation is of the form

$$ay'' + by' + cy = g(t)\,,$$

where a, b, and c are constants. Our guesses will depend on the form of $g(t)$. To find the general solutions, we must first solve the homogeneous equation, which is covered in the handout on second-order equations with constant coefficients. Then, we must find a particular solution to the nonhomogeneous equation. We'll use judicious guessing to find the particular solution, $\psi(t)$.

When g(t) is a Polynomial

Suppose we have a differential equation of the form

$$ay'' + by' + cy = a_0 + a_1 t + \cdots + a_n t^n\,.$$

In this case, $g(t)$ is a polynomial in t. We therefore guess that our particular solution, $\psi(t)$, is a polynomial as well of the same degree, n, with

$$\psi(t) = A_0 + A_1 t + \cdots + A_n t^n\,.$$

We can then compute $\psi'(t)$ and $\psi''(t)$ and plug it into our differential equation. Symbollically, this is rather ugly, so let's see it in some examples.

Example 1. Solve the differential equation $y'' + 2y' + y = t^2 + 1$.

We first solve the homogeneous equation, which is of the form $r^2 + 2r + 1 = (r+1)^2$. Our complementary solution is thus

$$y_c = c_1 e^{-t} + c_2 t e^{-t}.$$

Next, we note that the right hand side is a second degree polynomial, so our guess for $\psi(t)$ is $\psi(t) = A_0 + A_1 t + A_2 t^2$. Computing $\psi'(t)$ and $\psi''(t)$ and plugging into our original equation gives

$$\begin{aligned} 1 + t^2 &= \psi'' + 2\psi' + \psi \\ &= (2A_2) + 2(A_1 + 2A_2 t) + \left(A_0 + A_1 t + A_2 t^2\right) \\ &= (A_0 + 2A_1 + 2A_2) + (A_1 + 4A_2)t + (A_2)t^2. \end{aligned}$$

Matching coefficients with $1 + t^2$, we see that $A_2 = 1$. This tells us that $A_1 = -4$ by looking at the coefficient for t, which is equal to 0 in $1 + t^2$. Plugging in both of these to the constant term gives us $A_0 = 7$ in order to match with the constant term of 1 in $1 + t^2$. Thus, our particular solution is $\psi(t) = 7 - 4t + t^2$, and our general solution is

$$c_1 e^{-t} + c_2 t e^{-t} + 7 - 4t + t^2.$$

Example 2. Find a particular solution to $y'' + 2y' = 6t^2 - 2t + 2$.

In this problem, we're only concerned with finding $\psi(t)$. If, however, we guess $\psi(t) = A_0 + A_1 t + A_2 t^2$, then we have a problem when we plug in: there's no t^2 term on the left hand side! This is because we have only derivatives of $\psi(t)$. Thus, if we guess only a degree 2 polynomial, we can't get anywhere. Since the lowest-order derivative is the first, guessing a third degree polynomial will get us the desired t^2 term. We guess $\psi(t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3$ and proceed as before:

$$\begin{aligned} 6t^2 - 2t + 2 &= (2A_2 + 6A_3 t) + 2\left(A_1 + 2A_2 t + 3A_3 t^2\right) \\ &= (2A_1 + 2A_2) + (4A_2 + 6A_3)t + 6(A_3)t^2. \end{aligned}$$

This gives us $A_3 = 1$. Looking at the t term gives us $A_2 = -2$, and the constant term then gives us $A_1 = 3$. Notice how we could choose whatever value we want for A_0. This is because we only have derivatives of ψ, so the constant term always vanishes. We chose $A_0 = 0$, though any constant would work. This gives us a particular solution of

$$\psi(t) = 3t - 2t^2 + t^3.$$

From this, we can extrapolate that if $b = c = 0$ (i.e. we only have the second derivative), then we would guess a degree $n+2$ polynomial, which would have zero constant and linear terms. We summarize these three cases in the table on the first page.

When $\mathbf{g(t)}$ is a Polynomial Multiplied by $\mathbf{e^{\alpha t}}$

We now consider a differential equation of the form

$$ay'' + by' + cy = \left(a_0 + a_1 t + \cdots + a_n t^n\right) e^{\alpha t}.$$

First, solve the homogeneous equation $ay'' + by' + cy = 0$. Then, there are three cases, which we summarize in the table on the first page. Let's do an example.

Example 3. Find a particular solution to $y'' + y' - 2y = (-2 + 4t)e^{-t}$.

The homogeneous solution has the form $c_1 e^{-2t} + c_2 e^{t}$, so $e^{\alpha t}$ does not solve the homogeneous equation. We guess $\psi(t) = (A_0 + A_1 t)e^{-t}$. Using the Product Rule, we compute the derivatives of ψ:

$$\begin{aligned} \psi(t) &= (A_0 + A_1 t)e^{-t} \\ \psi'(t) &= A_1 e^{-t} - (A_0 + A_1 t)e^{-t} \\ \psi''(t) &= -2A_1 e^{-t} + (A_0 + A_1 t)e^{-t}. \end{aligned}$$

Next, we plug this into our equation:

$$\begin{aligned}(-2+4t)e^{-t} &= \left(-2A_1e^{-t}+(A_0+A_1t)e^{-t}\right)+\left(A_1e^{-t}-(A_0+A_1t)e^{-t}\right)-2\left((A_0+A_1t)e^{-t}\right)\\ &= (-2A_1+A_0+A_1t)e^{-t}+(A_1-A_0-A_1t)e^{-t}+(-2A_0-2A_1t)e^{-t}\,.\end{aligned}$$

We see that all of the terms have an e^{-t} (this will always happen), so we divide both sides by e^{-t} to get

$$\begin{aligned}-2+4t &= -2A_1+A_0+A_1t+A_1-A_0-A_1t-2A_0-2A_1t\\ &= (-2A_0-A_1)+(-2A_1)t\,,\end{aligned}$$

which gives $A_1=-2$ and $A_0=2$, so we have as our particular solution

$$\psi(t)=(2-2t)e^{-t}\,.$$

When $\mathbf{g(t)}$ has Trigonometric Functions

Suppose our equation is of the form

$$ay''+by'+cy=(a_0+\cdots+a_nt^n)\cos(\beta t)\,.$$

Recall that $e^{(\alpha+\beta i)t}=e^{\alpha t}\left(\cos(\beta t)+i\sin(\beta t)\right)$, so $\cos(\beta t)$ is the real part of $e^{\beta it}$. We then rewrite the equation as

$$ay''+by'+cy=\mathrm{Re}\left((a_0+\cdots a_nt^n)\,e^{\beta it}\right)$$

and solve as we did in the last section, with an exponent of $\alpha=\beta i$. Similarly, if we have

$$ay''+by'+cy=(a_0+\cdots+a_nt^n)\sin(\beta t)\,,$$

then we may use the fact that $\sin(\beta t)$ is the imaginary part of $e^{\beta it}$ to get

$$ay''+by'+cy=\mathrm{Im}\left((a_0+\cdots a_nt^n)\,e^{\beta it}\right)\,.$$

Let's see this in an example.

Example 4. Find a particular solution to $y''+y=\cos t$.

The homogeneous solution is $y_c(t)=c_1e^{-it}+c_2e^{it}$. We wish to guess $\psi(t)=e^{it}$. Now, e^{it} is one of the solutions, so by the previous section we guess $\psi(t)=A_0te^{it}$. We wish to solve the equation

$$y''+y=e^{it}\,,$$

so, we first compute

$$\begin{aligned}\psi' &= A_0e^{it}+A_0ite^{it}\\ \psi'' &= A_0ie^{it}+A_0ie^{it}-A_0te^{it}=-A_0te^{it}+(2A_0e^{it})i\,.\end{aligned}$$

Plugging in then gives

$$\begin{aligned}e^{it} &= \psi''+\psi\\ &= \left(-A_0te^{it}+2A_0ie^{it}\right)+\left(A_0te^{it}\right)\\ &= 2A_0ie^{it}\,.\end{aligned}$$

Thus, $2A_0i=1$, which gives us $A_0=\frac{1}{2i}=-\frac{1}{2}i$. This gives us

$$A_0te^{it}=\left(-\frac{1}{2}i\right)t(\cos t+i\sin t)=-\frac{1}{2}t\left(-\sin t+i\cos t\right)\,.$$

Recall that we originally wanted to solve $y''+y=\cos t$, so we must take the real part of this:

$$\psi(t)=\mathrm{Re}\left(-\frac{1}{2}t\left(-\sin t+i\cos t\right)\right)=-\frac{1}{2}t(-\sin t)=\frac{1}{2}t\sin t\,.$$

DISCLAIMER: This handout uses notation and methods from the textbook commonly used for M 427J courses taught at the University of Austin:
Braun, Martin, *Differential Equations and Their Applications*, 4th ed. Springer
December 5, 1992.