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Indeterminate Forms and L'Hôpital's Rule

Summary

Indeterminate forms are examples of limits that we cannot solve just by plugging in the x-limit into the function, but we can identify them by considering what happens to the function as x approaches the limit. The table below shows the 7 indeterminate forms.

Type	Form	Example
L'Hôpital's Rule	$\frac{0}{0}$	$\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{x-\frac{\pi}{2}}$
	$\pm\frac{\infty}{\infty}$	$\lim_{x\to\infty} \frac{x^2+1}{x^3-x+2}$
Product	$0\cdot(\pm\infty)$	$\lim_{x\to 0^+} x\ln x$
Difference	$\infty-\infty$	$\lim_{x\to 3^+} \sqrt{x^2-9}-x$
Power	0^0	$\lim_{x\to 0^+} (\sin x)^x$
	1^∞	$\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^{3x}$
	∞^0	$\lim_{x\to 0^+} (\ln x)^{2x}$

The first example is of the form $\frac{0}{0}$ since the numerator and denominator both approach 0 as $x\to\frac{\pi}{2}$. To give one more example, we see that, as $x\to 0^+$, $\ln x\to-\infty$, so that the third limit is of the form $0\cdot\infty$.

We will work out eaxmples of some of these indeterminate forms in the following pages: Examples 1 and 2 are of the form $\frac{0}{0}$, Example 3 is of the form $\frac{\infty}{\infty}$, Example 4 is of the form $0\cdot\infty$, Example 5 is of the form 0^0, and Example 6 is of the form 1^∞.

We use **L'Hôpital's Rule** to solve indeterminate forms, particularly the forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. It is important to remember that we can only use L'Hôpital's Rule on limits that are one of these two forms. Fortunately, any of the latter three forms listed above can be rewritten to be of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. L'Hôpital's Rule then states that if

$$\lim_{x\to c}\frac{f}{g}\to\frac{0}{0}\text{ or }\frac{\pm\infty}{\pm\infty},$$

then, assuming f' and g' exist,

$$\lim_{x\to c}\frac{f}{g}=\lim_{x\to c}\frac{f'}{g'}.$$

Be sure not to confuse this with the Quotient Rule for derivatives. Consider the first example in the table. Here $f(x)=\cos x$ and $g(x)=x-\frac{\pi}{2}$, both of which approach 0 as $x\to\frac{\pi}{2}$. Since $g'=1$ exists, we may use L'Hôpital's Rule to get

$$\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{x-\frac{\pi}{2}}=\lim_{x\to\frac{\pi}{2}}\frac{-\sin x}{1}=\frac{-1}{1}=-1.$$

We may need to use L'Hôpital's Rule more than once in a problem. We can use L'Hôpital's Rule repeatedly, so long as each subsequent limit is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Example 3 illustrates this. Examples 4, 5, and 6 show how to go from other indeterminate forms to $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Sometimes students identify certain forms such as $\frac{0}{\infty}$ or $\frac{\infty}{0}$ as indeterminate, when in fact these forms are determinate. We list these in the last section. These two forms give 0 and ∞ respectively.

Indeterminate Forms

Often when taking limits, we cannot just evaluate the function at the desired x-value. Consider, for example, the limit of the function $f(x)=x\ln x$ as x approaches 0 from the right:

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}x\ln x.$$

On the one hand, the x goes to 0, but on the other hand the $\ln x$ approaches $-\infty$. This gives us our first **indeterminate form**, $0\cdot\infty$ (we ignore the negative sign). This form is called "indeterminate" because it's not clear whether the x "wins out" and brings the limit to 0, or if the tendency of $\ln x$ towards $-\infty$ causes the limit to diverge.

There are many different types of indeterminate forms, which we list at the end of this section and in the Summary on the first page. For now, let us consider two more types of indeterminate forms: 0^0 and $\frac{0}{0}$. For the former, consider the two functions $f(x) = x^0$ and $g(x) = 0^x$. For (positive) nonzero x, we know that $x^0 = 1$ and $0^x = 0$. On the other hand, 0^0 is not defined. Consider the limit of each function as x approaches 0:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^0 \quad \text{and} \quad \lim_{x\to 0^+} g(x) = \lim_{x\to 0} 0^x .$$

Note that 0^x is not defined for negative x (otherwise we would have $\frac{1}{0}$), so we can only approach it from the right. We see that both limits are of the form 0^0 in the sense that both the base and the exponent either approach 0 or *are* 0. It turns out that the limit for $f(x)$ is 1 and the limit for $g(x)$ is 0. This example shows why we call such limits "indeterminate"; though they are of the same form, 0^0, the two limits are different. Later on in this handout, we will compute the limit of x^x as $x \to 0^+$, which is also of the form 0^0.

Now let's turn our attention to $\frac{0}{0}$. Consider the two rational functions

$$f(x) = \frac{x^3 + x^2 - x - 1}{x^2 - x} \quad \text{and} \quad g(x) = \frac{\sin x}{x} ,$$

and consider the following limits of each function:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^3 + x^2 - x - 1}{x^2 - x} \quad \text{and} \quad \lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{\sin x}{x} .$$

Both limits are of the form $\frac{0}{0}$ since the numerators and denominators of both all approach 0. On the other hand, we could use L'Hôpital's Rule to show that the limit for $f(x)$ is 4 and that the limit for $g(x)$ is 1. $\frac{0}{0}$ is thus also an indeterminate form since the two limits are not equal.

There are 7 types of indeterminate forms: $\frac{0}{0}, \pm\frac{\infty}{\infty}, 0\cdot(\pm\infty), \infty - \infty, 0^0, 1^\infty$, and ∞^0. In general, when dividing by 0 or ∞, raising 0 or ∞ to a power, or using 0 or ∞ as an exponent, an indeterminate form might pop up. In some cases, such as $\frac{0}{\infty}$, the limit will be unambiguous, but other times there will be an indeterminate form!

L'Hôpital's Rule

We sometimes find limits that are indeterminate forms and which we can also manipulate algebraically. For example,

$$\lim_{x\to 1} \frac{x^2 - 1}{x^2 - x}, \quad \text{and} \quad \lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4}$$

are both of the form $\frac{0}{0}$ and can be solved by factoring and multiplying by conjugates respectively. On the other hand, without L'Hôpital's Rule, we don't have a simple way to evaluate limits such as

$$\lim_{x\to\infty} \frac{\ln x}{x}, \quad \text{or} \quad \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}} .$$

L'Hôpital's Rule allows us to evaluate limits that take one of the seven indeterminate forms discussed above. To be precise, we can only use L'Hôpital's Rule on the forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$, but each of the other five forms can actually be rewritten to take the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. We will see how to do this in some later examples. For now, note that the function $f(x) = x\ln x$, which we saw was of the form $0\cdot\infty$ as $x \to 0^+$, can be rewritten as $f(x) = \frac{\ln x}{1/x}$, which now has the form $\frac{\infty}{\infty}$ as $x \to 0^+$.

L'Hôpital's Rule states that if

$$\lim_{x\to c} \frac{f(x)}{g(x)} \to \frac{0}{0}, \text{ or } \frac{\pm\infty}{\pm\infty} ,$$

then, assuming $f'(x)$ and $g'(x)$ exist,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} .$$

Keep in mind that this is not the Quotient Rule; we are simply taking the derivative of the numerator over the derivative of the denominator. If it turns out that $\lim_{x\to c} \frac{f'}{g'}$ is also of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, may we simply use L'Hôpital's Rule again, assuming that f'' and g'' exist. In the following examples we will see why this is so useful. Remember, however, to try and evaluate the limit at each step.

Example 1. Evaluate $\lim_{x\to 0} \frac{\sin x}{x}$ using L Hôpital's Rule.

First note that as $x \to 0$, $N = \sin x$ and $D = x$ both approach 0. We can see this by plugging in 0 to the functions $\sin x$ and x. This limit is therefore of the form $\frac{0}{0}$, so we may use L'Hôpital's Rule. $N' = \cos x$, and $D' = 1$. Thus, we have

$$\lim_{x\to 0} \frac{N}{D} = \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \lim_{x\to 0} \frac{N'}{D'}\,.$$

We may now evaluate $\lim_{x\to 0} \frac{\cos x}{1}$ by simply plugging in $x = 0$. This gives us our answer

$$\lim_{x\to 0} \frac{\sin x}{x} = 1\,.$$

This fact is sometimes given without proof. This is how to prove it!

Example 2. Evaluate $\lim_{x\to 1} \frac{x^3+x^2-x-1}{x^2-x}$.

As we saw in the previous section, this is of the form $\frac{0}{0}$. Thus, we may use L'Hôpital's Rule. Taking the derivative of the numerator and of the denominator gives

$$\lim_{x\to 1} \frac{x^3+x^2-x-1}{x^2-x} = \lim_{x\to 1} \frac{3x^2+2x-1}{2x-1} = \frac{4}{1} = 4\,.$$

After each step of L'Hôpital's Rule, remember to evaluate the new limit to see if it is no longer an indeterminate form.

Example 3. Evaluate $\lim_{x\to\infty} \frac{(\ln x)^2}{x}$.

Note that this is of the form $\frac{\infty}{\infty}$. We use L'Hôpital's Rule to get

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{(2\ln x)\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln x}{x}\,.$$

We see that this is again of the form $\frac{\infty}{\infty}$. Thus, we use L'Hôpital's Rule again to get

$$\lim_{x\to\infty} \frac{2\ln x}{x} = \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = 0\,.$$

This gives us our answer, $\lim_{x\to\infty} \frac{(\ln x)^2}{x} = 0$. Remember that we can use L'Hôpital's Rule as many times as necessary, so long as the new limit is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 4. Evaluate $\lim_{x\to 0^+} x\ln x$.

Right now, this limit is of the form $0\cdot\infty$, so we cannot use L'Hôpital's Rule yet, since it's not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. At the beginning of this section, we noted that $x\ln x = \frac{\ln x}{1/x}$, and that rewriting it this way gets the function into the desired $\frac{\infty}{\infty}$ form. We may now use L'Hôpital's Rule to obtain

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}\,.$$

If we plug $x = 0$ into this last expression, we still get $\frac{\infty}{\infty}$, so one might think that we ought to use L'Hôpital's Rule again. This would not solve the problem—in fact, we could take the derivative of the numerator and of the denominator indefinitely and get $\frac{\infty}{\infty}$ (for practice, verify this). Thankfully, we can rewrite $\frac{1/x}{-1/x^2}$ as just $-x$, so the limit then becomes

$$\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0\,.$$

This implies $\lim_{x\to 0^+} x\ln x = 0$. Remember to try and evaluate the new limit each time before using L'Hôpital's Rule. We see from this example that we can rewrite other indeterminate forms as $\frac{\infty}{\infty}$ in order to evaluate them using L'Hôpital's Rule.

Example 5. Evaluate $\lim_{x\to 0^+} x^x$.

This limit is indeterminate of the form 0^0. Nevertheless, we may solve it using L'Hôpital's Rule. First note that $x = e^{\ln x}$. We may then write

$$x^x = \left(e^{\ln x}\right)^x = e^{x\ln x}\,.$$

We saw $x\ln x$ in the last example. As $x \to 0^+$, $x\ln x \to 0$. We then have

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = \lim_{x\to 0^+} e^0 = 1\,.$$

Technically, we didn't use L'Hôpital's Rule in this Example, but that's only because we had already used it to do the work of evaluating $x\ln x$. This trick of rewriting x^x as $e^{x\ln x}$ is an important trick to keep in mind for evaluating limits of the form $0^0, \infty^0$, and 1^∞. Since expressions of the form $e^{-\infty}, e^0$, and e^∞ always work out to $0, 1$, and ∞ respectively, rewriting the indeterminate forms as $e^{f(x)}$ allows us to focus only on the limit $f(x)$. Here, for example, we rewrote x^x as $e^{f(x)}$, where $f(x) = x\ln x$. We then used the limit of $x\ln x$ to get our final limit.

Example 6. Evaluate $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$.

This limit takes the indeterminate form 1^∞, since $\frac{1}{x} \to 0$ as $x \to \infty$. Recall the trick we used in the previous example. We may rewrite $1 + \frac{1}{x}$ as $e^{\ln(1+1/x)}$, so that we get

$$\left(1 + \frac{1}{x}\right)^x = \left(e^{\ln(1+1/x)}\right)^x = e^{x\ln(1+1/x)}\,.$$

This expression is a bit scary, but remember that now we just need to focus on the exponent, $x\ln(1 + \frac{1}{x})$. If we take the limit as $x \to \infty$, we see that this expression is of the form $\infty \cdot 0$ since $\ln 1 = 0$. To handle this indeterminate form, we must rewrite it as

$$x\ln\left(1 + \frac{1}{x}\right) = \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}\,.$$

Taking the limit as $x \to \infty$, we see that the numerator approaches $\ln 1 = 0$, and that the denominator approaches 0. Thus, our limit is of the form $\frac{0}{0}$, and so we may now use L'Hôpital's Rule to obtain

$$\lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\left(\frac{-1/x^2}{1+1/x}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1 + \frac{1}{x}}\,.$$

The $\frac{1}{x}$ in the denominator approaches 0 as $x \to \infty$, so that the final limit is $\frac{1}{1} = 1$. We now conclude that the limit of $x\ln\left(1 + \frac{1}{x}\right)$ as $x \to \infty$ is 1. Using this in our original problem, we have that

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln(1+1/x)} = e^1 = e\,.$$

This limit is in fact the definition of e, but this work verifies that the definition is consistent with the properties of limits and with L'Hôpital's Rule.

Things that are Not Indeterminate Forms

Students sometimes incorrectly identify certain limits as indeterminate forms. Here we'll cover some common mistakes of this type. Here we list these forms and give their limits:

Form	**Limit**	**Example**
$\frac{k}{\infty}$	0	$\lim_{x\to\infty} \frac{2}{x}$
$\frac{\infty}{k}$	∞	$\lim_{x\to\infty} \frac{x}{\pi}$
$\frac{k}{0}$	∞	$\lim_{x\to 0} \frac{2}{x}$
$\frac{\infty}{0}$	∞	$\lim_{x\to\infty} \frac{\ln x}{1/x}$
$\frac{0}{\infty}$	0	$\lim_{x\to 0} \frac{x}{\ln x}$

In the first three forms, k represents any constant such as $2, \pi$, or e. For the form $\frac{\infty}{0}$, the numerator is already increasing without bound, and the shrinking denominator further makes the limit tend towards ∞. For the last form, $\frac{0}{\infty}$, the numerator is shrinking and the denominator is growing without bound, both of which make the limit tend towards 0.