**What is a Limit?**

Understanding the limits of functions gives us insight into the graph of that function. Informally, the **limit of a function f(x) as x approaches c**, written $\lim_{x \to c} f(x)$, is the value $f(x)$ approaches as x gets "close to c" from both sides, though we do not care what happens *at $x = c$*. We can also consider the **left-hand limit** $\lim_{x \to c^-} f(x)$ and the **right-hand limit** $\lim_{x \to c^+} f(x)$, as we approach $x = c$ only from the left or only from the right respectively. The "$-$" and "$+$" to the *right* of c denote this concept, rather than negativity or positivity, which would be $-c$ or $+c$. The delta-epsilon definition for the limit of a function makes this notion of "close to" precise, but this is definition too technical for our purposes. Let's illustrate the limit concept with an example.
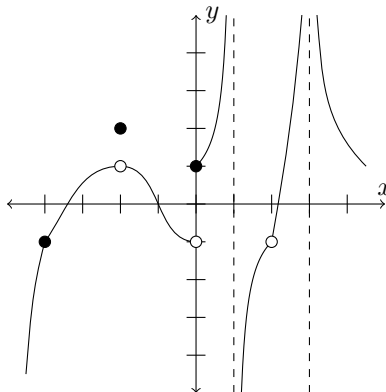
In the plot above, we have labelled several points. We will compute the limit of $f(x)$ at $x = -4$, $x = -2$, $x = 0$, $x = 1$, $x = 2$, and $x = 3$ by inspecting the graph of $f(x)$ above.

- First, at $x = -4$, we see that $f(x)$ approaches -1 from both sides, so

$$\lim_{x \to -4} f(x) = -1 \quad \left(= \lim_{x \to -4^-} f(x) = \lim_{x \to -4^+} f(x) \right).$$

- What about at $x = -2$? Well, even though the function is 2 *at $x = 2$*, $f(x)$ *approaches* 1 *as x approaches* -2. Thus, the limit as $x \to -2$ of $f(x)$ is 1 rather than 2. This is a very important concept: **the limit of f(x) as x $\to$ c is not necessarily equal to f(x)**. At $x = -4$, this was the case, but as we've just seen this is not the case at $x = -2$.

- Next, consider $x = 0$. From the left, i.e. for x-values less than but close to 0, $f(x)$ approaches -1. On the other hand, from the right, i.e. for x-values greater than but close to 0, $f(x)$ approaches 1. This gives us

$$\lim_{x \to 0^-} f(x) = -1 \quad \text{and} \quad \lim_{x \to 0^+} f(x) = 1.$$

  What about $\lim_{x \to 0} f(x)$, the limit itself? Remember, the fact that the function value is 1 at $x = 0$ *does not* mean that $\lim_{x \to 0} f(x)$ equals 1. In fact, since the left-hand and right-hand limits disagree, we say that **the limit does not exist (DNE)**. The limit does not exist when the left- and right-hand limits disagree.

- At $x = 1$, we have a vertical asymptote. From the left, the x-values approach very large positive values without bound. In such a case, we say that

$$\lim_{x \to 1^-} f(x) = \infty.$$

  This equals "$=$" sign has a slightly different use here than before, since ∞ is not a number per se, but rather this notation is just meant to convey the idea that $f(x)$ increases without bound as $x \to 1$

from the left. From the right, the x-values approach very large (in absolute value) negative numbers. Similar to the case before, we write

$$\lim_{x \to 1^+} f(x) = -\infty \,.$$

Again, the equals sign does not mean equality in the traditional sense. It only denotes that $f(x)$ decreases without bound as $x \to 1$ from the right. Now, the left-hand and right-hand limits do not agree, so $\lim_{x \to 1} f(x)$ does not exist.

- At $x = 2$, we immediately see that $\lim_{x \to 2} f(x) = -1$. Even though $f(x)$ is not defined at $x = 1$, we only care what happens at x-values *close to* 1, but not at 1 itself.

- Finally, at $x = 3$, we see that both the left-hand and right-hand limits are positive ∞. Some textbooks say that the limit does not exist because ∞ is not a number.

  Though ∞ is not a number, the left- and right-hand limits agree, so we might say, unlike before, that $\lim_{x \to 3} f(x) = \infty$. This statement refers to the behavior of the function at $x = 3$ rather than asserts that the limit is a single number: the function f increases without bound as x approaches 3 from the left or from the right. This disagreement is merely a matter of convention rather than an argument over mathematical fact.

Few functions have all of these features. We constructed this example to show the several different cases to consider when taking limits of functions.

Continuity

Informally, we can determine that a function is continuous if we can draw its graph without picking up the pencil. This is not a rigorous method, but it helps one visualize continuity. Like the limit, continuity has a formal delta-epsilon definition that makes the notion precise, but we will not cover it here. We can do better than the "pencil method", however, by using limits. A function $f(x)$ is **continuous at x = c** if

$$\lim_{x \to c} f(x) = f(c) \,.$$

This property actually requires 3 things:

1. $f(x)$ must be defined at $x = c$. This is usually the simplest to check.

2. $\lim_{x \to c} f(x)$ must exist.

3. The limit $\lim_{x \to c} f(x)$ and the function value $f(c)$ must *agree*, i.e. must be equal.

If $f(x)$ fails *any* of these three conditions at $x = c$, then we say that $f(x)$ is **discontinuous (or not continuous) at x = c**. Let's use these three conditions to determine the continuity of the function on page 1.

- $f(x)$ is continuous at $x = -4$. By inspection: (1) $f(c)$ exists, (2) $\lim_{x \to c} f(x)$ exists, and (3) $\lim_{x \to c} f(x) = f(c)$.

- $f(x)$ is discontinuous at $x = -2$ because the limit and the function value do not agree.

- $f(x)$ is discontinuous at $x = 0$ because the limit of $f(x)$ does not exist, as we saw above.

- $f(x)$ is discontinuous at $x = 1$ because it is not defined at $x = 2$. It fails conditions (2) and (3) as well.

- $f(x)$ is discontinuous at $x = 2$ since it is not defined there.

- $f(x)$ is discontinuous at $x = 3$ because it is not defined there.

Determining condition (3) requires evaluating $\lim_{x \to c} f(x)$. We cover how to do this without using the graph of the function in the handout "Evaluating Limits". Below is a table with examples of common functions that are continuous (on their domains):

Type	Examples
Polynomials	1; $x+2$; $x^3 - 4x + 1$
Trigonometric functions	$\sin x$; $\cos x$; $\tan x$ (if $\cos x \neq 0$)
Exponential functions	e^x; 2^{-x}
Sums of cont. functions	$x^2 + \cos x$; $e^x - \sin x$
Products of cont. functions	$x\left(e^x + \cos x\right)$; $x^2 \cos x$
Quotients of cont. (denominator $\neq 0$)	$\frac{x^2}{x+1}$, $(x \neq -1)$; $\frac{2}{e^x}$ (for all x)
Compositions of cont. functions	$\cos\left(x^2\right)$; $e^{\sin(x+2)}$