Limits and Convergence of Sequences

What is a Sequence?

Formally, a **sequence** is a function whose domain is the set of positive integers: 1, 2, . . ., and we denote a sequence by $\{a_n\}$. We can also denote a sequence by the formula for its terms, as we do now. Examples of sequences include $a_n = 2n$ and $b_n = n^2 + 1$. Then we have

$$\begin{aligned} &a_1 = 2(1) = 2, && a_2 = 2(2) = 4, && a_3 = 2(3) = 6\,, \\ &b_1 = (1)^2 + 1 = 2, && b_2 = (2)^2 + 1 = 5, && b_3 = (3)^2 + 1 = 10\,. \end{aligned}$$

Here, we see that a_n and b_n are both functions of n, where n is a positive integer. Rather than writing $a_n(n)$ or $b_n(n)$ (as we would do for $f(x)$), we abbreviate to a_n. Another notation for the above sequences, which we'll call **list notation**, is

$$\begin{aligned} \{a_n\} &= 2, 4, 6, \ldots \\ \{b_n\} &= 2, 5, 10, \ldots\,. \end{aligned}$$

There's no fixed rule for how many terms in the sequence should be included with this list notation, and it would have been valid to write $\{a_n\} = 2, 4, 6, 8, \ldots$ as well. The ellipses at the end of both sequences in this notation signify that the sequence continues for infinitely many values of n.

n is called the **index** of the sequence. We sometimes refer to a_1 as the "first term", a_2 as the "second term", and so on. This is why it's convenient for our indices to start at 1, but we could have just as well started with $n = 0$. The "n^{th} term" refers to the general form of the sequence. For a_n, the n^{th} term is $2n$, and for b_n the n^{th} term is $n^2 + 1$. Sometimes it is convenient to include the n^{th} term in the list notation for the sequence, like so:

$$\begin{aligned} \{a_n\} &= 2, 4, 6, \ldots, 2n, \ldots \\ \{b_n\} &= 2, 5, 10, \ldots, n^2 + 1, \ldots\,. \end{aligned}$$

This compact form gives the formula for computing any desired term as well as the first few terms. The ellipses after the n^{th} term show that the sequence continues on indefinitely. When we want to ignore terms in between two specific terms in the sequence, we also use ellipses. If we wanted to emphasize the first three terms of a sequence and then, say, the 20^{th} and 21^{st} terms, we would write:

$$\begin{aligned} \{a_n\} &= 2, 4, 6, \ldots, 40, 42, \ldots \\ \{b_n\} &= 2, 5, 10, \ldots, 401, 442, \ldots\,. \end{aligned}$$

Just as familiar functions like $\cos x$ and e^x have limits, sequences also have limits. As we shall see, finding the limit of a sequence is very similar to evaluating the limits of functions.

Limits and Convergence of Sequences

The formal definition for the limit of a sequence is quite technical and not usually taught in freshman Calculus, so we will omit it. Informally, the **limit** of a sequence $\{a_n\}$ is L if a_n approaches L as n gets larger. We write this as

$$\lim_{n\to\infty} a_n = L\,.$$

Note that this is very similar to the notation $\lim_{x\to c} f(x)$ for a function x when c is any real number. For sequences, however, it only makes sense to take limits as $n \to \infty$. Thinking of limits of sequences as like limits of functions helps in finding the limit of a particular sequence, assuming that function has a limit as $n \to \infty$. For example, the sequences

$$\begin{aligned} a_n &= \frac{1}{n}\,, \\ b_n &= \frac{n^2 + 2n - 1}{3n^2 + 1}\,, \\ c_n &= \frac{2n}{n^2 + 1} \end{aligned}$$

have limits

$$\lim_{n\to\infty} a_n = 0\,,$$
$$\lim_{n\to\infty} b_n = \frac{1}{3}\,,$$
$$\lim_{n\to\infty} c_n = 0\,,$$

which can be seen by taking the limit of each sequence as $n \to \infty$. For instance, $\frac{1}{n} \to 0$ as $n \to \infty$, which is essentially the same as the function $f(x) = \frac{1}{x}$ going to 0 as $x \to \infty$. The point of this discussion is that taking limits of certain sequences is like taking limits of functions in Differential Calculus.

A sequence is **convergent** if it has a limit. Otherwise, the sequence is **divergent**. For a convergent sequence $\{a_n\}$ with a limit L, we say that $\{a_n\}$ **converges** to L. A convergent sequence will always have only one limit. Before we do examples, let's go over some helpful properties for the limits of sequences.

Properties of the Limits of Sequences

Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences with $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then

1. $\lim\limits_{n\to\infty} (ka_n) = k \lim\limits_{n\to\infty} a_n = kA$, where k is a constant,
2. $\lim\limits_{n\to\infty} (a \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = A \pm B$,
3. $\lim\limits_{n\to\infty} (a_n b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n = AB$,
4. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \dfrac{A}{B}$, if $B \neq 0$ and $b_n \neq 0$ for all n, and
5. if a_n is in the domain of f and if f is continuous for all n, then $\lim\limits_{n\to\infty} f(a_n) = f(A)$.

These properties are very similar to the properties for limits of functions. They are useful for transforming difficult limits into simpler ones, as we shall see in some of the following examples.

Example 1. Find the limit, if it exists, of the sequence $a_n = \frac{3}{n}$.

We immediately have $\lim_{n\to\infty} \frac{3}{n} = 0$. This sequence converges with limit 0.

Example 2. Find the limit, if it exists, of the sequence $a_n = \sin n$.

The limit $\lim_{n\to\infty} \sin n$ does not exist. This is because the sine function oscillates between -1 and 1 indefinitely and never approaches a fixed number. Thus, this sequence diverges.

Example 3. Find the limit, if it exists, of the sequence $a_n = (-1)^n$.

This sequence also has no limit. Again, this is because it oscillates between -1 and 1; in fact, it switches between those two values, so it never approaches a fixed number. This sequence diverges.

Example 4. Find the limit, if it exists, of the sequence $a_n = \frac{\ln n}{2n}$.

We need to compute $\lim_{n\to\infty} \frac{\ln n}{2n}$. We do this using L'Hôpital's Rule since this limit is of the form $\frac{\infty}{\infty}$:

$$\lim_{n\to\infty} a_n = \lim n \to \infty \frac{\ln n}{2n} = \frac{1/n}{2} = \lim_{n\to\infty} \frac{1}{2n} = 0\,.$$

Since limits of sequences are like limits of functions, we can also use theorems like L'Hôpital's Rule.

Example 5. Find the limit, if it exists of the sequence $a_n = \left(\frac{n^2+1}{3n^2-5}\right)^3$.

The limit as $n \to \infty$ of the inside function, $\frac{n^2+1}{3n^2-5}$, is $\frac{1}{3}$. Our sequence is the cube of this inside function. The function $f(x) = x^3$ is a continuous function, and we can view a_n as f applied to $\frac{n^2+1}{3n^2-5}$. By Property 5, we have

$$\lim_{n\to\infty} a_n = \left(\frac{1}{3}\right)^3 .$$