Linear Independence, Spanning Sets, and Bases

Vector Spaces and Linear Combinations of Vectors

Vector spaces are a special type of structure with elements called **vectors** that can added together and scaled by **scalars**. Furthermore, these spaces come with a special **zero vector**, which we'll denote $\mathbf{0}$, with the property that $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$. These vectors and scalars have to respect certain axioms, which we won't list here. In this handout, we'll typically write scalars normally and vectors in **boldface**. $\mathbb{R}^2$ and $\mathbb{R}^3$ are examples of vector spaces. In these spaces, the vectors are the ordered pairs or ordered triples respectively of the form

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{or} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

where the components are real numbers. The scalars in both of these spaces are the real numbers. We add and scale these vectors componentwise. As an example, we calculate in $\mathbb{R}^2$:

$$3\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

There are many examples of vector spaces, even beyond those which we can write in the familiar coordinate form above. In this handout, we'll just focus on $\mathbb{R}^2$ and $\mathbb{R}^3$ in our examples. In the example above, we scaled two vectors by different scalars and then added the results together. Such a process is called taking a linear combination of the two vectors. In any vector space, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are vectors, then a **linear combination** of $\mathbf{v}_1, \ldots \mathbf{v}_n$ is an expression of the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n,$$

where $a_1, \ldots, a_n$ are scalars. In the example above, we had a linear combination of two vectors with $a_1 = 3$ and $a_2 = -2$.

Linear Independence

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is **linearly independent** if no single vector can be written as a linear combination of the other vectors. Otherwise, the vectors are **linearly dependent**. Let's examine what this means through examples, starting with two vectors in $\mathbb{R}^2$. Consider the vectors

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \text{ and } \mathbf{y} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Observe that $\mathbf{y} = 2\mathbf{x}$. $\mathbf{y}$ is thus a linear combination of $\mathbf{x}$–a "linear combination" of one vector is just that vector multiplied by a scalar. Thus, $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. On the other hand, suppose we also have

$$\mathbf{z} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

We quickly see that $\mathbf{x}$ and $\mathbf{z}$ are linearly independent, since they are not scalar multiples of each other. Similarly, $\mathbf{y}$ and $\mathbf{z}$ are linearly independent. If we consider all three vectors as the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$, however, this set *is* linearly dependent since $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. In particular, we could write $\mathbf{y} = 2\mathbf{x} + 0\mathbf{z}$ so that $\mathbf{y}$ is a linear combination of the other vectors $\mathbf{x}$ and $\mathbf{z}$.

Let's consider another set of three vectors, this time in $\mathbb{R}^3$:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad \text{and } \mathbf{z} = \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix}.$$

None of these three vectors is a scalar multiple of one of the others. We can work out, however, that these three vectors are linearly dependent since $\mathbf{z}$ is actually a linear combination of $\mathbf{x}$ and $\mathbf{y}$. One can check that $\mathbf{z} = 2\mathbf{x} + \mathbf{y}$.

It may seem that we pulled this linear combination out of thin air. To address this issue, we now introduce a logically equivalent definition of linear independence that is useful for computation: a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is **linearly independent** if the equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$$

has no nontrivial solution (where we are solving for $a_1, \ldots, a_n$). In other words, only the trivial solution $a_1 = \cdots = a_n = 0$ solves the above equation. Using the same $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ as in the preceding example, let's show that $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ are linearly dependent. Since we are working in $\mathbb{R}^3$, the zero vector $\mathbf{0}$ is just the vector with 0 in all three components. We wish to solve the equation

$$a_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + a_3 \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

for a_1, a_2, and a_3. We can combine the left-hand side into a single vector, each of whose components must equal 0. This gets us the linear system

$$\begin{aligned} a_1 + 2a_2 + 4a_3 &= 0 \\ a_1 - \ a_2 \ + \ a_3 \ &= 0 \\ 2a_1 + 3a_2 + 7a_3 &= 0\,. \end{aligned}$$

We can solve this in a variety of ways. We can use the second line to get $a_1 = a_2 - a_3$. Plugging this information into lines 1 and 3 gets

$$\begin{aligned} (a_2 - a_3) + 2a_2 + 4a_3 = 3a_2 + 3a_3 &= 0 \\ 2(a_2 - a_3) + 3a_2 + 7a_3 = 5a_2 + 5a_3 &= 0\,, \end{aligned}$$

which we can then solve to get $a_2 = -a_3$, for any a_2 we choose. For convenience, let's choose $a_2 = 1$. Then $a_1 = 1 - (-1) = 2$. Thus, we have the nontrivial solution $a_1 = 2, a_2 = 1$, and $a_3 = -1$. Since we have a nontrivial solution to $a_1\mathbf{x} + a_2\mathbf{y} + a_3\mathbf{z} = \mathbf{0}$, the three vectors are linearly dependent. In particular $2\mathbf{x} + \mathbf{y} - \mathbf{z} = \mathbf{0}$. Note that if we add $\mathbf{z}$ to both sides, we get the equation $2\mathbf{x} + \mathbf{y} = \mathbf{z}$ that we saw previously.

For $\mathbb{R}^2$, $\mathbb{R}^3$, and in general $\mathbb{R}^n$, checking for linear independence boils down to solving a system of linear equations. Let's do a couple more examples.

Example 1. Determine whether $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are linearly independent, where

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{z} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

We must determine whether

$$a_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has a nontrivial solution, so we must solve the linear system of equations

$$\begin{aligned} a_1 + 2a_2 + \ a_3 \ &= 0 \\ 2a_1 + 3a_2 + 3a_3 &= 0\,. \end{aligned}$$

Using $a_1 = -2a_2 - a_3$ and plugging into the second equation, we get

$$2(-2a_2 - a_3) + 3a_2 + 3a_3 = -a_2 + a_3 = 0\,.$$

This equation has infinitely many solutions since for any a_2 we can set a_3 equal to a_2. Once we have a_2 and a_3, we get a_1 since $a_1 = -2a_2 - a_3$. We can therefore find a nontrivial solution with $a_2 = 1$ so that $a_3 = 1$ and $a_1 = -3$. The three vectors are therefore linearly dependent. One can quickly verify that $-3\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$.

Example 2. Determine whether $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are linearly independent, where

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

As usual, we must determine whether

$$a_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has any nontrivial solutions. This gets us the linear system

$$\begin{aligned} a_1 \quad + \quad a_3 &= 0 \\ a_2 + a_3 &= 0 \\ a_1 + a_2 \qquad &= 0\,. \end{aligned}$$

The first equation gives $a_3 = -a_1$. Plugging this into the second equation yields $-a_1 + a_2 = 0$. We then have

$$\begin{aligned} -a_1 + a_2 &= 0 \\ a_1 + a_2 &= 0\,, \end{aligned}$$

which, by adding the two equations together, gives $a_1 = a_2 = 0$. Then $a_3 = 0$ as well. We have shown that $a_1 = a_2 = a_3 = 0$, so we do not have a nontrivial solution to $a_1\mathbf{x} + a_2\mathbf{y} + a_3\mathbf{z} = \mathbf{0}$; hence $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ are linearly independent.

Spanning Sets

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in a vector space V **spans** V if every vector $\mathbf{w}$ in V can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, i.e. if

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{w}\,,$$

has a solution for every $\mathbf{w}$ in V. Notice the similarity to the condition we were checking for linear independence. In that case, we were checking the special case $\mathbf{w} = \mathbf{0}$ and making note of whether there was only the trivial solution or if there were nontrivial solutions. Now, we are letting $\mathbf{w}$ be an arbitrary vector and trying to determine whether we can always find $a_1, \ldots, a_n$ that make the above equation true. Let's illustrate this through examples.

Example 3. Determine whether $\mathbf{x}$ and $\mathbf{y}$ span $\mathbb{R}^2$, where

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We want to solve the equation

$$a_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where w_1 and w_2 are two fixed, arbitrary real numbers (this is how we would write $\mathbf{w}$ in $\mathbb{R}^2$). Writing the left-hand side as a single vector and matching components, we get the system of equations

$$\begin{aligned} a_1 + a_2 &= w_1 \\ -a_1 + a_2 &= w_2 \end{aligned}$$

where w_1 and w_2 are fixed. Does this have a solution? This boils down to solving for a_1 and a_2 in terms of w_1 and w_2. Adding the two equations together and dividing both sides by 2 gives $a_2 = \frac{1}{2}(w_1 + w_2)$.

Subtracting the second equation from the first and dividing by 2 gives $a_1 = \frac{1}{2}(w_1 - w_2)$. Thus, we do have a solution!

How does this help us in practice? Suppose we had $\mathbf{w} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$. Then

$$a_1 = \frac{w_1 - w_2}{2} = \frac{3-(-5)}{2} = 4\,, \quad \text{and} \quad a_2 = \frac{w_1 + w_2}{2} = \frac{3+(-5)}{2} = -1\,.$$

Using these coefficients, we indeed see that $a_1\mathbf{x} + a_2\mathbf{y} = 4\mathbf{x} - \mathbf{y} = \mathbf{w}$. In general, we have shown that for any $\mathbf{w}$ we chose in $\mathbb{R}^2$, we could always find a_1 and a_2 satisfying $a_1\mathbf{x} + a_2\mathbf{y} = \mathbf{w}$.

Example 4. Determine whether $\mathbf{x}$ and $\mathbf{y}$ span $\mathbb{R}^3$, where

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

As before, we fix an arbitrary vector $\mathbf{w}$ and check whether

$$a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

has any solutions. This gives us the system of equations

$$\begin{aligned} a_1 &= w_1 \\ a_2 &= w_2 \\ a_2 &= w_3\,. \end{aligned}$$

This system, however, does not always have a solution. In fact, if we ever have $w_2 \neq w_3$, then we can't find a suitable a_1 or a_2, since we could not satisfy the last two equations. Thus, these two vectors do *not* span $\mathbb{R}^3$.

Example 5. Determine whether $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ span $\mathbb{R}^2$, where

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We could check this using the same condition that we've been using, but we don't have to. We already saw in Example 3 that $\mathbf{x}$ and $\mathbf{y}$ span $\mathbb{R}^2$. Adding the vector $\mathbf{z}$ into our list of vectors does not suddenly make the vectors not span anymore. To see this, let $\mathbf{w}$ be any vector in $\mathbb{R}^2$. Since $\mathbf{x}$ and $\mathbf{y}$ already span $\mathbb{R}^2$, we can find a_1 and a_2 so that $a_1\mathbf{x} + a_2\mathbf{y} = \mathbf{w}$. This automatically means that the equation

$$a_1\mathbf{x} + a_2\mathbf{y} + a_3\mathbf{z} = \mathbf{w}$$

has a solution, since we can set $a_3 = 0$. This shows that adding vectors to a spanning set will not change whether or not that set is spanning. In this sense, the vector $\mathbf{z}$ is "redundant".

Bases and Dimension

So far we have considered sets of vectors and certain properties about them, specifically whether they are linearly independent and whether they are spanning. In Example 5, we saw that we can have "redundant" vectors we considering spanning sets. In that particular case, we could've removed the vector $\mathbf{z}$ and we still would've had a spanning set. If we had then removed $\mathbf{y}$ as well, however, we would've no longer had a spanning set.

This leads to the question of how we can tell whether a spanning set of vectors has any "redundant" vectors. It turns out (we will not prove it here) that a spanning set has redundant vectors exactly when it is a linearly dependent set of vectors. In Example 5, you can check that $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ are linearly dependent.

You can also check that the spanning set of vectors $\mathbf{x}$ and $\mathbf{y}$ in Example 3 was also linearly independent, i.e. there were no "redundant" vectors.

A **basis** is a set of vectors that is both linearly independent and spanning, and it captures this notion of a non-redundant spanning set. To check whether or not a set of vectors is a basis, we check for both linear independence and whether or not that set is spanning. If both are satisfied, then that set is a basis. Try completing these next two problems as an exercise:

Exercise 1. Show that the set of vectors in Example 3 forms a basis for $\mathbb{R}^2$.

Exercise 2. Show that the set of vectors in Example 2 forms a basis for $\mathbb{R}^3$.

Any given vector space can have infinitely many bases. The **standard basis** for $\mathbb{R}^2$ is the set of vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and the standard basis for $\mathbb{R}^3$ is similar, but contains three vectors with a 1 in one of the coordinates and a 0 in the other coordinates. These are particularly nice bases because it is easy to make any vector $\mathbf{w}$ from them in their respective vector spaces. For any $\mathbf{w} \in \mathbb{R}^2$, for example, we have $\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2$. For higher dimensions $\mathbb{R}^n$, the standard basis is similar, with n vectors.

Any vector space has a basis. One fact about bases is that for any given vector space V, any basis for V will always have the same number of vectors. Thus, any basis for $\mathbb{R}^2$ will have 2 vectors, and any basis for $\mathbb{R}^3$ will have 3 vectors. Because of this fact, it makes sense to define the **dimension** of a vector space, which is the number of vectors in a basis for that vector space. $\mathbb{R}^2$ has dimension 2 and $\mathbb{R}^3$ has dimension 3, which we often write $\dim \mathbb{R}^2 = 2$ and $\dim \mathbb{R}^3 = 3$. In general, $\dim \mathbb{R}^n = n$.