

What is a Max-min Problem?

Max-min problems, also known as optimization problems, are word problems that ask for the maximum or minimum values of a function, often according to a certain constraint. Here are two examples:

1. A farmer wants to build a rectangular fence that will enclose 120 square feet of area for a couple chickens. Two opposite sides are to be made of wood at a cost of \$5 per foot. The other two sides are to be made of wire at a cost of \$6 per foot. What are the dimensions of the fence that will cost him the least to make? Assume that each side of the fence must be at least 3 feet.
2. A race starts on an island 4 miles offshore in a straight river. The finishing point is on the shore, 6 miles down the river. Bill can swim at a rate of 3 mph and run at a rate of 6 mph. What route should he take to win the race? Assume the current is negligible.

In Question 1, we want to minimize the cost of the fence, subject to the constraint that it encloses 120 square feet of area for his dog. In Question 2, we want to choose a route that minimizes the time it takes to get from start to finish. Unlike in Question 1, there isn't a simple constraint in this problem. Nevertheless, it is still a max-min problem.

These problems usually require taking the derivative of a certain function. Recall that a function $f(x)$ has critical points whenever $f'(x) = 0$ or when $f'(x)$ is undefined. Minima and maxima only occur at these critical points and at the endpoints of the function's domain, so once we find these critical points we can check the function at each of these points to find its absolute maximum or minimum. We now give a step-by-step method for solving max-min problems and then use this method to solve the two examples above.

Steps for a Max-min Problem

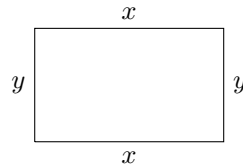
1. Read the problem carefully and identify the quantity to be optimized (minimized/maximized).
2. Draw a diagram of the problem, labelling relevant constants and variables.
3. Determine your **objective function**, the function to be maximized or minimized.
4. Write down any constraint equations, i.e. equations representing any extra conditions that must be satisfied. The problem might not have extra constraints.
5. Rewrite the objective function in terms of only one variable. This is often accomplished by manipulating constraint equations and substituting into the objective function.
6. Determine the domain of the objective function.
7. Take the first derivative of the objective function and determine where the critical values are, i.e. find where the derivative is equal to 0 or is undefined.
8. Check the objective function at the critical values and the endpoints to find extrema. Then use this to answer the question.

Let's use these steps to solve Questions 1 and 2.

Solving Question 1

1. In this problem, we wish to minimize, or optimize, the cost of the fence.
2. There are two variables in the problem: the height and width of the rectangular area to be fenced off. Let's call the width x and the height y . Remember that the wood and the wire fences cost two different amounts. We can arbitrarily choose which one is x and which one is y . Let x be the wood fence, and let y be the wire fence. We now draw the farmer's plot of land:

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To reiterate, the top and bottom sides are made of wood, and the left and right sides are made of wire.

3. We want to minimize the cost of the fencing for this plot of land, so our objective function is the cost of the fence, let's call it $C(x, y)$. The wood fence (x) costs \$5 per foot, and the wire fence (y) costs \$6 per foot. Since we have two sides of wood fence, the total length of wood fence we use is $2x$. Similarly, the total amount of wire fence we use is $2y$. The cost equation is then

$$C(x, y) = 5(2x) + 6(2y) = 10x + 12y.$$

4. To minimize this function, we need to find its critical points. In order to do that, we need to have cost in terms of one variable. We can choose to express it as either x or y . We'll express the cost $C(x)$ as a function of x . To do this, we need to get y in terms of x , so we now turn to our constraint equation. Recall that we want the fenced-in area to be 120 square feet. The area of the plot is given by $A(x, y) = xy$, and we want the $A(x, y) = 120$. Thus, we have

$$120 = xy.$$

Dividing both sides by x gives us

$$y = \frac{120}{x}.$$

5. Let's plug this information into our objective function $C(x, y)$:

$$C(x, y) = 10x + 12\left(\frac{120}{x}\right) = 10x + \frac{1,440}{x}.$$

We now have the cost in terms of one variable:

$$C(x) = 10x + \frac{1,440}{x}.$$

6. Let's now consider the domain of our objective function. By this we mean the range of possible values for x (or y , depending on whether we choose to write the cost in terms of x or in terms of y). We know that $x \geq 3$. On the other hand, we must have $y \geq 3$ as well, so we cannot make x as large as we like. For example, if $x = 60$, then $120 = xy$ gives $y = 2$, which is too small. Plugging $y = 3$ into $120 = xy$ tells us that $x \leq 40$. Our domain is thus $3 \leq x \leq 40$.
7. We compute the derivative of $C(x)$:

$$C'(x) = 10 - \frac{1,440}{x^2}.$$

We see immediately that $C'(x)$ is not defined at $x = 0$, though 0 is not in our domain. $C'(x)$ is defined everywhere else, so we now set it equal to 0 and solve for x :

$$\begin{aligned} 0 &= 10 - \frac{1,440}{x^2} \\ \frac{1,440}{x^2} &= 10 \\ 1,440 &= 10x^2 \\ x^2 &= 144 \\ x &= \pm 12, \end{aligned}$$

and we can eliminate $x = -12$ since we cannot have a negative length. Thus, we have $x = 12$ as our one critical point.

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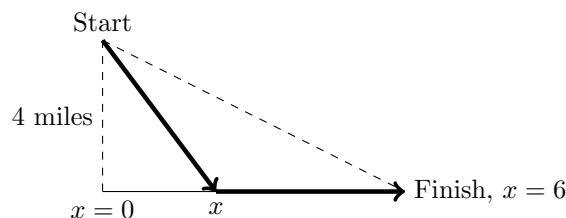
8. We must check $x = 3, x = 12$, and $x = 40$. We could use the Second Derivative Test to see that $C(x)$ has a relative minimum at $x = 12$. However, we must check $C(x)$ at the endpoints to determine whether $x = 12$ is an *absolute* minimum. It is helpful to write this in a table:

x , feet	$C(x)$, \$
3	510
12	240
40	760

From this, we see that there is an absolute minimum at $x = 12$. Note that we wouldn't need to use the Second Derivative Test to see whether $x = 12$ is a minimum, as we can read this off of the table of critical points and endpoints. Question 1 specifically asks for the dimensions of the fence that will cost the farmer the least to make. We must now find y . We plug $x = 12$ into our constraint equation $120 = xy$ and quickly find that $y = 10$. Thus, the dimensions of the fence are $x = 12$ and $y = 10$.

Solving Question 2

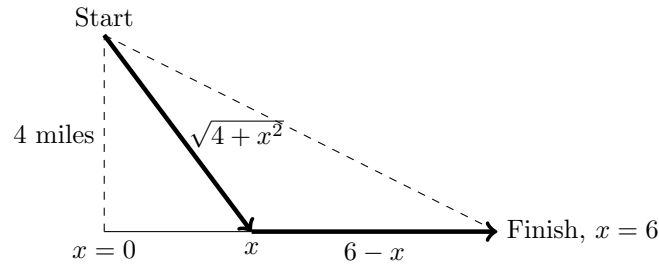
1. We want to find the route that will win Bill the race, so we wish to minimize the time it takes for him to get to the finish. One might think that he should just swim directly to the finish line, but he might be able to save time if he swims to the shore and then runs to the finish since he runs more quickly than he swims.
2. There are a couple different ways we could choose our variables. We could let x_s and x_r be the distance that Bill swims and runs respectively. This over-complicates things, however, since we can just let x be the point at which Bill reaches the shore. This is because choosing our point on the shore determines the route for the race: Bill should swim straight to the chosen point on the shore and then run from there straight to the finish line. This x ranges from $x = 0$, the point on the shore closest to Bill, to $x = 6$, which is the finish line. Let's draw this in a diagram:



The dashed edges represent the “extrema paths” of the problem: if Bill started at a negative x value (i.e. to the left of $x = 0$), he would definitely not be taking the shortest path since he would have to travel unnecessary distance on land. Similarly, anything beyond $x = 6$ is certainly not optimal.

3. We wish to minimize the time Bill takes to get to the finish. Let's call this function $T(x)$, where x is the point at which Bill gets to shore. There are two parts to Bill's route: the swimming part, and the running part. To get equations for these in terms of x , let's add to our diagram. Bill swims along the hypotenuse of a right triangle, and we can calculate the length of this hypotenuse using the Pythagorean Theorem: it has length $\sqrt{4 + x^2}$. The length Bill must run along the shore is $6 - x$. We display this on our diagram (we enlarge it for clarity):

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The question tells us that Bill swims at a rate of 3 mph and runs at a rate of 6 mph. Using $\text{Speed} = \frac{\text{Distance}}{\text{Time}}$ (i.e. $\text{Time} = \frac{\text{Distance}}{\text{Speed}}$), we have that Bill spends $\frac{1}{3}\sqrt{4+x^2}$ hours in the river and $\frac{1}{6}(6-x)$ hours on land. Putting this together, we get

$$T(x) = \frac{1}{3}\sqrt{4+x^2} + \frac{1}{6}(6-x).$$

4. This problem does not have any extra constraints.
5. This function is already in terms of one variable, so we don't need to do any additional work on this step.
6. The domain of our objective function is $0 \leq x \leq 6$. If $x < 0$, we immediately see that the route is not optimal, since Bill should never move to the left. We also immediately see that $x > 6$ is not optimal since Bill would need to run back to the left in that case.
7. We now find the critical points of $T(x)$. We compute

$$T'(x) = \frac{1}{3} \left(\frac{2x}{2\sqrt{4+x^2}} \right) + \frac{1}{6}(-1) = \frac{x}{3\sqrt{4+x^2}} - \frac{1}{6}.$$

This derivative is defined everywhere in our domain. We now set $T'(x) = 0$ and solve for x :

$$\begin{aligned} 0 &= \frac{x}{3\sqrt{4+x^2}} - \frac{1}{6} \\ \frac{1}{6} &= \frac{x}{3\sqrt{4+x^2}} \\ \frac{1}{36} &= \frac{x^2}{9(4+x^2)} \\ \frac{1}{4}(4+x^2) &= x^2 \\ 1 + \frac{1}{4}x^2 &= x^2 \\ \frac{3}{4}x^2 &= 1 \\ x^2 &= \frac{4}{3} \\ x &= \pm \frac{2}{\sqrt{3}}. \end{aligned}$$

We throw out $-\frac{2}{\sqrt{3}}$. Our one critical point is at $x = \frac{2}{\sqrt{3}}$. We can use a calculator to evaluate $T(x)$ at this critical point and at the endpoints $x = 0$ and $x = 6$. This gives us the following table:

x, miles	T(x), hours
0	1.67
$\frac{2}{\sqrt{3}}$	1.58
6	2.92

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We conclude that there is an absolute minimum at $x = \frac{2}{\sqrt{3}}$.

8. We now answer the question completely: Bill should swim to a point on the shore $\frac{2}{\sqrt{3}}$ (about 1.15) miles downstream from the starting point and then run to the finish from there.

Summary

Above we listed the steps for solving optimization problems. The numbering of the steps isn't so important, but it *is* important to remember the general flow of optimization problems:

- Read the problem, determine the variables and constants, and draw a diagram of the situation.
- From the diagram, determine the **objective function**—that is, find an equation for the quantity to be optimized.
- Write down equations for any given or implicit **constraints**.
- Write the objective function in terms of **one variable** (if it is not in that form already). This often uses the constraint equations.
- Identify the **domain** of the objective function, keeping in mind the endpoints of the domain (if there are any).
- Find the **critical points** of the one-variable objective function, i.e. where its derivative is zero or undefined.
- **Evaluate the objective function** at the critical points AND at the endpoints. Students often forget to evaluate the function at the endpoints. List the data in a table and determine the smallest or largest (depending on the problem) value for the objective function. It is important to evaluate the function at the endpoints because sometimes the optimal value occurs at one of the endpoints.
- Use this information to **answer the problem completely**, for example by determining what the other variable needs to be.