Rolle’s Theorem and the Mean Value Theorem

Rolle’s Theorem

Before we state the Mean Value Theorem, we will state a special case of it: Rolle’s Theorem. **Rolle’s Theorem** is as follows: if \( f(x) \) is a function satisfying

1. \( f(x) \) is continuous on the closed interval \([a, b]\),
2. \( f(x) \) is differentiable on the open interval \((a, b)\), and
3. \( f(a) = f(b) \),

then there is a number \( c \) with \( a < c < b \) such that

\[
f'(c) = 0.
\]

Consider the function \( f(x) = 4 - x^2 \) on \([-1, 1]\). What does Rolle’s Theorem say about this situation? Since \( f(x) \) is continuous and differentiable everywhere, it is certainly continuous on the closed interval \([-1, 1]\) and differentiable on the open interval \((-1, 1)\) (here, \( a = -1 \) and \( b = 1 \)). We also have that \( f(-1) = f(1) = 3 \). Rolle’s Theorem then says that there is a number \( c \) with \(-1 < c < 1\) such that \( f'(c) = 0 \). In other words, the function \( 4 - x^2 \) has a critical point somewhere between \(-1\) and 1.

In fact, we can find this value \( c \). We know that \( f'(x) = -2x \). We want to check when \( f'(x) = -2x = 0 \). This occurs at \( x = 0 \), which is between \( x = -1 \) and \( x = 1 \), so here \( c = 0 \). Below on the left, we graph \( f(x) = 4 - x^2 \) (solid) and show the tangent line (dashed) at \( c = 0 \), which has slope 0.

Most problems will ask for the value of \( c \) satisfying the conclusion of Rolle’s Theorem, which amounts to finding where \( f'(x) = 0 \) on the open interval \((a, b)\). Rolle’s Theorem states that there has to be at least one such \( c \) so long as \( f(x) \) satisfies the hypotheses.

One deeper insight from Rolle’s Theorem is the following: if a continuous and differentiable function \( f(x) \) has \( n \) roots, then the derivative \( f'(x) \) has at least \( n - 1 \) roots. Imagine the \( x \)-values for the roots of \( f(x) \) lined up on the \( x \)-axis. For every two consecutive roots, Rolle’s Theorem tells us that \( f'(x) = 0 \) between those two roots (since \( f(x) = 0 \) at both of the two roots). So, if \( f(x) \) has 3 distinct roots, then \( f'(x) = 0 \) somewhere between the first and the second roots and also somewhere between the second and third roots. In the right-hand figure above, we show this, with dashed lines representing the tangent lines where \( f'(x) = 0 \). There are two such points, each between two of the roots.

The Mean Value Theorem

The **Mean Value Theorem** is as follows: if \( f(x) \) is a function such that

1. \( f(x) \) is continuous on the closed interval \([a, b]\), and
2. \( f(x) \) is differentiable on the open interval \((a, b)\),

then there is a value \( c \) with \( a < c < b \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]
First, note that when \( f(a) = f(b) \), the numerator is 0, which gives Rolle’s Theorem. The expression on the right-hand side of the equation gives the slope of the line between \((a, f(a))\) and \((b, f(b))\). The Mean Value Theorem then says that there is a point \( c \) between \( a \) and \( b \) such that the tangent line at \( x = c \) has the same slope as the secant line connecting \((a, f(a))\) and \((b, f(b))\), i.e. the tangent line at \( x = c \) is parallel to the secant line.

**Example 1.** Find all values of \( c \) satisfying the conclusion of the Mean Value Theorem for \( f(x) = \frac{1}{4}x^3 + 1 \) on the interval \([-2, 2]\).

First, note that \( f(x) \) is continuous and differentiable everywhere, in particular it is continuous on the closed interval \([-2, 2]\) and differentiable on the open interval \((-2, 2)\). We wish to find a value \( c \) such that

\[
f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{3 - (-1)}{4} = 1.
\]

So, we compute \( f'(x) = \frac{3}{4}x^2 \) and set it equal to 1. We then solve for \( x \):

\[
\frac{3}{4}x^2 = 1
\]

\[
x^2 = \frac{4}{3}
\]

\[
x = \pm \frac{2}{\sqrt{3}}.
\]

Both of these values lie in the open interval \((-2, 2)\), so we have \( c = \pm \frac{2}{\sqrt{3}} \). We graph \( f(x) \) along with the secant line (dotted) and the two lines of tangency (dashed):