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Rolle's Theorem and the Mean Value Theorem

Rolle's Theorem

Before we state the Mean Value Theorem, we will state a special case of it: Rolle's Theorem. **Rolle's Theorem** is as follows: if $f(x)$ is a function satisfying

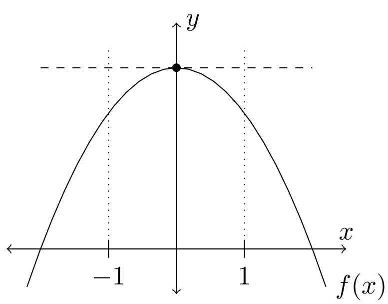
1. $f(x)$ is continuous on the closed interval $[a, b]$,
2. $f(x)$ is differentiable on the open interval (a, b), and
3. $f(a) = f(b)$,

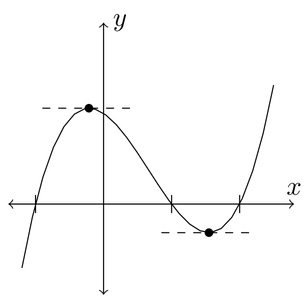
then there is a number c with $a < c < b$ such that

$$f'(c) = 0\,.$$

Consider the function $f(x) = 4 - x^2$ on $[-1, 1]$. What does Rolle's Theorem say about this situation? Since $f(x)$ is continuous and differentiable everywhere, it is certainly continuous on the closed interval $[-1, 1]$ and differentiable on the open interval $(-1, 1)$ (here, $a = -1$ and $b = 1$). We also have that $f(-1) = f(1) = 3$. Rolle's Theorem then says that there is a number c with $-1 < c < 1$ such that $f'(c) = 0$. In other words, the function $4 - x^2$ has a critical point somewhere between -1 and 1.

In fact, we can find this value c. We know that $f'(x) = -2x$. We want to check when $f'(x) = -2x = 0$. This occurs at $x = 0$, which is between $x = -1$ and $x = 1$, so here $c = 0$. Below on the left, we graph $f(x) = 4 - x^2$ (solid) and show the tangent line (dashed) at $c = 0$, which has slope 0.





Most problems will ask for the value of c satisfying the conclusion of Rolle's Theorem, which amounts to finding where $f'(x) = 0$ on the open interval (a, b). Rolle's Theorem states that there has to be at least one such c so long as $f(x)$ satisfies the hypotheses.

One deeper insight from Rolle's Theorem is the following: *if a continuous and differentiable function $f(x)$ has n roots, then the derivative $f'(x)$ has* at least *$n - 1$ roots.* Imagine the x-values for the roots of $f(x)$ lined up on the x-axis. For every two consecutive roots, Rolle's Theorem tells us that $f'(x) = 0$ between those two roots (since $f(x) = 0$ at both of the two roots). So, if $f(x)$ has 3 distinct roots, then $f'(x) = 0$ somewhere between the first and the second roots and also somewhere between the second and third roots. In the right-hand figure above, we show this, with dashed lines representing the tangent lines where $f'(x) = 0$. There are two such points, each between two of the roots.

The Mean Value Theorem

The **Mean Value Theorem** is as follows: if $f(x)$ is a function such that

1. $f(x)$ is continuous on the closed interval $[a, b]$, and
2. $f(x)$ is differentiable on the open interval (a, b),

then there is a value c with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

First, note that when $f(a) = f(b)$, the numerator is 0, which gives Rolle's Theorem. The expression on the right-hand side of the equation gives the slope of the line between $(a, f(a))$ and $(b, f(b))$. The Mean Value Theorem then says that there is a point c between a and b such that the tangent line at $x = c$ has the same slope as the secant line connecting $(a, f(a))$ and $(b, f(b))$, i.e. the tangent line at $x = c$ is parallel to the secant line.

Example 1. Find all values of c satisfying the conclusion of the Mean Value Theorem for $f(x) = \frac{1}{4}x^3 + 1$ on the interval $[-2, 2]$.

First, note that $f(x)$ is continuous and differentiable everywhere, in particular it is continuous on the closed interval $[-2, 2]$ and differentiable on the open interval $(-2, 2)$. We wish to find a value c such that

$$f'(c) = \frac{f(2) - f(-2)}{(2) - (-2)} = \frac{(3) - (-1)}{4} = 1\,.$$

So, we compute $f'(x) = \frac{3}{4}x^2$ and set it equal to 1. We then solve for x:

$$\begin{aligned} \frac{3}{4}x^2 &= 1 \\ x^2 &= \frac{4}{3} \\ x &= \pm\frac{2}{\sqrt{3}}\,. \end{aligned}$$

Both of these values lie in the open interval $(-2, 2)$, so we have $c = \pm\frac{2}{\sqrt{3}}$. We graph $f(x)$ along with the secant line (dotted) and the two lines of tangency (dashed):