Non-Linear Inequalities

Introduction

In this handout, we discuss solving and graphing inequalities involving certain non-linear functions, namely polynomials and rational functions.

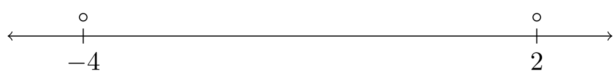
Quadratic Inequalities

Example 1. Solve the inequality $x^2 + 2x - 8 < 0$.

We first want to find where the quadratic expression *equals* 0. We can do this via factoring:

$$x^2 + 2x - 8 = 0 \longrightarrow (x+4)(x-2) = 0,$$

which tells us that the expression equals 0 when $x = -4$ or $x = 2$. Next, we'll draw a number line and label the points where $x^2 + 2x - 8$ is 0:



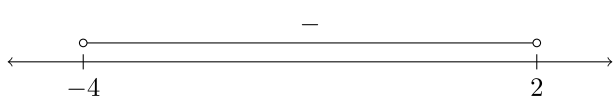
This divides the number line into three intervals: $(-\infty, -4), (-4, 2)$, and $(2, \infty)$. A quadratic equation can only change whether it's positive or negative at its zeros. Thus, on each of these three intervals, the $x^2 + 2x - 8$ is either always positive or always negative. We can determine which case we're dealing with by evaluating the quadratic expression at one point in each interval.

We could plug x-values into x^2+2x-8, but it is actually easier to work with the factored form $(x+4)(x-2)$. We show this in a table:

x-values	$(x+4)$	$(x-2)$	$(x+4)(x-2)$
$x = -5$	$-$	$-$	$+$
$x = 0$	$+$	$-$	$-$
$x = 3$	$+$	$+$	$+$

Here, we use an x-value within each of the intervals: $x = -5$ is in the interval $(-\infty, -4)$, and so on. $(x+4) < 0$ and $(x-2) < 0$ when $x = -5$, so their product $(x+4)(x-2)$ is positive when $x = -5$.

The inequality $x^2 + 2x - 8 < 0$ holds when there is a "$-$" in the last column, which here occurs on the interval containing $x = 0$, namely the interval $(-4, 2)$. We show our final answer on a number line:



$(-4, 2)$

Example 2. Solve for x when $-3x^2 + 8x \leq -x^2 - 6x + 20$.

Now we have quadratic expressions on either side of the inequality. Let's collect all the terms on the right-hand side:

$$-3x^2 + 8x \leq -x^2 - 6x + 20 \longrightarrow 0 \leq 2x^2 - 14x + 20.$$

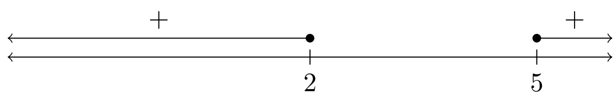
Now we can use our method from the previous example, finding where the quadratic expression equals 0. We can divide both sides by 2 to get

$$0 \leq x^2 - 7x + 10 = (x-5)(x-2).$$

As before, this divides our number line into three intervals: $(-\infty, 2), (2, 5)$, and $(5, \infty)$. We find an x-value in each interval and determine the sign of $(x-5)(x-2)$:

x-values	$(x-2)$	$(x-5)$	$(x-2)(x-5)$
$x=0$	$-$	$-$	$+$
$x=3$	$+$	$-$	$-$
$x=6$	$+$	$+$	$+$

Recall that we wish to solve $(x-5)(x-2) \geq 0$. This is when our rightmost column is positive, i.e. on the intervals $(-\infty, 2)$ and $(5, \infty)$. We graph this on our number line:



$(-\infty, 2] \cup [5, \infty)$

Polynomial Inequalities

We now generalize to polynomials of degree greater than 2. As before, we wish to find the roots of our polynomial. This will split our number line into intervals, and we test points on each interval to determine whether our polynomial is positive or negative on that interval.

Example 3. Solve the inequality $x^3 + 4x^2 + 4x \leq 0$.

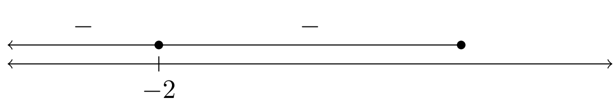
It turns out we can factor this polynomial:

$$x^3 + 4x^2 + 4x = x\left(x^2 + 4x + 4\right) = x(x+2)^2$$

This splits our number line into three intervals: $(-\infty, -2), (-2, 0)$, and $(0, \infty)$. We make our table as before:

x-values	x	$(x+2)$	$x(x+2)^2$
$x=-3$	$-$	$-$	$-$
$x=-1$	$-$	$+$	$-$
$x=1$	$+$	$+$	$+$

Note that $(x+2)^2$ is always positive, so the sign of the $(x+2)$ factor at each x-value is actually irrelevant. We are looking for x-values where the polynomial is negative. Reading from the table, this is when x is in either $(-\infty, -2)$ or $(-2, 0)$. Since our inequality is inclusive, we include the points -2 and 0 as well, giving us our final number line plot:



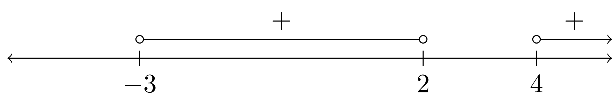
$(-\infty, 0]$

Example 4. Solve for x when $(x-2)(x-4)(x+3)^3 > 0$.

In general it is difficult to factor a degree 3 or higher polynomial. This degree 5 polynomial is already factored for us. It splits our number line into four intervals: $(-\infty, -3), (-3, 2), (2, 4)$, and $(4, \infty)$. As usual, we refer to a table:

x-values	$(x-2)$	$(x-4)$	$(x+3)$	$(x-2)(x-4)(x+3)^3$
$x=-4$	$-$	$-$	$-$	$-$
$x=0$	$-$	$-$	$+$	$+$
$x=3$	$+$	$-$	$+$	$-$
$x=5$	$+$	$+$	$+$	$+$

We are looking for an x-value where our polynomial is strictly positive. Reading this from our table, we get the solution:



$$(-3,2)\cup(4,\infty)$$

Inequalities with Rational Functions

We will focus on rational functions of polynomials, using similar methods as before. Remember that we cannot divide by 0, so we will now consider x-values that make the denominator equal to 0 in addition to those that make the numerator equal to 0.

Example 5. Solve for x when $\frac{2}{x-2} \geq \frac{1}{x+5}$.

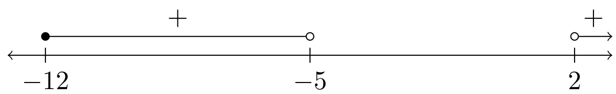
As before, we wish to collect all of our terms on one side of the inequality and have 0 on the other side. In addition, we want to have our function as one fraction. We do this now:

$$\begin{aligned}
\frac{2}{x-2} &\geq \frac{1}{x+5} \\
\frac{2}{x-2} - \frac{1}{x+5} &\geq 0 && \text{Collect terms on one side} \\
\frac{2(x+5)-1(x-2)}{(x-2)(x+5)} &\geq 0 && \text{Get a common denominator} \\
\frac{2x+10-x+2}{(x-2)(x+5)} &\geq 0 \\
\frac{x+12}{(x-2)(x+5)} &\geq 0.
\end{aligned}$$

The numerator equals 0 when $x=-12$, and the denominator equals 0 when $x=-5$ or when $x=2$. We make our familiar table:

x-values	$(x+12)$	$(x+5)$	$(x-2)$	$\frac{x+12}{(x-2)(x+5)}$
$x=-13$	$-$	$-$	$-$	$-$
$x=-6$	$+$	$-$	$-$	$+$
$x=0$	$+$	$+$	$-$	$-$
$x=3$	$+$	$+$	$+$	$+$

We graph this solution on our number line:



$$[-12, -5) \cup (2, \infty)$$

Note that the circles at -5 and 2 are open, since these are points where the denominator is 0, hence where neither of the functions in the original inequality is defined.