

## When to Use Partial Fraction Decomposition

We use **partial fraction decomposition (PFD)** to solve certain integrals. For example, we can integrate  $\int \frac{1}{(x-1)(x+1)} dx$  by using PFD to determine that

$$\frac{1}{(x-1)(x+1)} = \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1},$$

which we can then integrate using  $u$ -substitution. We will use PFD when our integrands are rational functions (fractions with polynomials in the numerator and in the denominator) whose denominator can be factored. If we have say,  $\frac{x^2+1}{(x-1)(x+3)(x-2)}$ , then our goal will be to find expressions  $K(x)$ ,  $L(x)$ , and  $M(x)$  such that

$$\frac{x^2+1}{(x-1)(x+3)(x-2)} = \frac{K(x)}{x-1} + \frac{L(x)}{x+3} + \frac{M(x)}{x-2}.$$

In this particular case, the denominator is factored into distinct linear factors, and we will see that  $K$ ,  $L$ , and  $M$  will be constants.

## The Method of PFD: Unique Linear Factors

We will teach the method through various examples. Let's say we want to do the integral  $\int \frac{1}{x^2-4} dx$ . This is a prime candidate for PFD because the denominator is a factorable polynomial. We factor  $x^2-4$  as  $(x-2)(x+2)$ . Here we have the case of **unique linear factors**: they are **unique** because they are distinct, and they are **linear** because the highest exponent of each factor is 1. When our denominator factors as distinct linear factors, we do the decomposition as follows:

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2},$$

where  $A$  and  $B$  are constants. We must solve for  $A$  and  $B$ , and afterwards we can perform the integral. To solve for  $A$  and  $B$ , we first put the right-hand side over a common denominator:

$$\frac{1}{x^2-4} = \frac{A}{x-2} \cdot \frac{x+2}{x+2} + \frac{B}{x+2} \cdot \frac{x-2}{x-2} = \frac{A(x+2) + B(x-2)}{(x-2)(x+2)}.$$

For this equality to hold, we must have the numerators be equal for all  $x$ :

$$1 = A(x+2) + B(x-2).$$

Again, this must hold for all  $x$ . We can solve for  $A$  and  $B$  via two methods:

1. Firstly, we can use the fact that this equality holds for all  $x$ . In particular, it holds for  $x = 2$  and  $x = -2$  (we chose these  $x$ -values so that the coefficients of  $B$  and  $A$  respectively vanish).  $x = 2$  gives  $1 = 4A$ , and  $x = -2$  gives  $1 = -4B$ . From this, we conclude that  $A = \frac{1}{4}$  and  $B = -\frac{1}{4}$ .
2. On the other hand, we can distribute  $A$  and  $B$  to get

$$1 = Ax + 2A + Bx - 2B = (A+B)x + (2A-2B).$$

For equality to hold, we must have  $A+B=0$  since the left-hand side has no  $x$ -terms. We must also have  $1=2A-2B$ . This is a system of linear equations, which we know how to solve from high school Algebra. For example, we can use the first equation to get  $B=-A$ . Then  $1=2A+2A=4A$ , so  $A=\frac{1}{4}$  and thus  $B=-\frac{1}{4}$ .

Either of these methods yields  $A = \frac{1}{4}$  and  $B = -\frac{1}{4}$ . In the first method, we plug in convenient  $x$ -values to find  $A$  and  $B$ . In the second method, we match the coefficients of the right-hand side to those of the left-hand side to get a system of linear equations from which we can get  $A$  and  $B$ .

# Partial Fraction Decomposition

Sometimes the first method won't completely solve the problem, but the second method will *always* work. Even when it won't completely solve the problem, however, the first method is still useful. It's important to be able to use both methods effectively.

To sum up our work, we have

$$\frac{1}{x^2 - 4} = \frac{\frac{1}{4}}{x - 2} + \frac{-\frac{1}{4}}{x + 2}.$$

We can now integrate this using familiar methods (for example, with  $u$ -substitution), to get

$$\int \frac{1}{x^2 - 4} dx = \int \frac{\frac{1}{4}}{x - 2} + \frac{-\frac{1}{4}}{x + 2} dx = \frac{1}{4} \ln|x - 2| - \frac{1}{4} \ln|x + 2| + C.$$

Before we move onto the next section, keep in mind that it does not matter whether we have  $A$  over  $x - 2$  as we did or if we choose to have  $A$  over  $x + 2$  and  $B$  over  $x - 2$ . We will get different  $A$  and  $B$  values, but they will simply be swapped, and our final integral will not change. It *is* critical, however, that we keep the fractions consistent. In the end, we still have  $A$  over  $x - 2$  and  $B$  over  $x + 2$  since that was what we chose in the beginning.

## The Method of PFD: Repeated Linear Factors

What if the denominator does not factor into unique linear factors? Suppose, for example, that we want to find the integral  $\int \frac{x^2 + 2x}{(x - 2)^3} dx$ . Here, we have the case of a **repeated linear factor**: the linear factor  $x - 2$  has an exponent of 3. We start our decomposition as follows:

$$\frac{x^2 + 2x}{(x - 2)^3} = \frac{A}{x - 2} + \frac{B}{(x - 2)^2} + \frac{C}{(x - 2)^3},$$

where  $A$ ,  $B$ , and  $C$  are constants. Notice that the denominators are all the distinct powers of  $x - 2$  up to 3. If the exponent of the denominator were 5, we would have two additional terms with  $(x - 2)^4$  and  $(x - 2)^5$  in the denominators. Now, we get the right-hand side over a common denominator as before:

$$\frac{x^2 + 2x}{(x - 2)^3} = \frac{A(x - 2)^2 + B(x - 2) + C}{(x - 2)^3}.$$

This gives us

$$x^2 + 2x = A(x - 2)^2 + B(x - 2) + C.$$

Let's try the first method. Plugging in  $x = 2$  gives  $8 = C$  since the  $A$  and  $B$  terms cancel. Plugging in  $C = 8$  and moving it over to the left-hand side, we now have

$$x^2 + 2x - 8 = A(x - 2)^2 + B(x - 2).$$

If we try to plug in  $x = 2$  we cancel both the  $A$  and the  $B$  terms, so we must now resort to the second method. Distributing out the terms on the right-hand side and collecting like-terms, we get

$$x^2 + 2x - 8 = Ax^2 + (-4A + B)x + (4A - 2B),$$

which immediately gives us  $A = 1$  since the coefficients of the  $x^2$ -terms must be equal. Then plugging in  $A = 1$  into either of  $2 = -4A + B$  or  $-8 = 4A - 2B$  gives  $B = 6$ . Putting this back into our original integral and using  $u$ -substitution gives

$$\begin{aligned} \int \frac{x^2 + 2x}{(x - 2)^3} dx &= \int \frac{1}{x - 2} + \frac{6}{(x - 2)^2} + \frac{8}{(x - 2)^3} dx \\ &= \ln|x - 2| - \frac{6}{x - 2} - \frac{4}{(x - 2)^2} + C. \end{aligned}$$

## The Method of PFD: Irreducible Quadratic Factors and Other Types

# Partial Fraction Decomposition

Some of the factors in the denominator may be irreducible **quadratic (or higher power) factors**, i.e. factors whose highest exponents are 2 or higher but cannot be factored into linear factors. Examples of irreducible quadratic factors include  $x^2 + 1$  and  $3x^2 - 2x - 2$ . Suppose we wish to find a partial fraction decomposition of  $\frac{x^3}{(x^2+1)(3x^2-2x-2)}$ . We decompose it as follows:

$$\frac{x^3}{(x^2 + 1)(3x^2 - 2x - 2)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{3x^2 - 2x - 2},$$

where  $A, B, C$ , and  $D$  are constants. Notice how the numerators now have  $x$ -terms, whereas before they were constants.

Just as we could have repeated linear factors, we can have **repeated irreducible quadratic factors** as well. We handle them similarly. For example:

$$\frac{x}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2}.$$

Once we have this form, the method of PFD proceeds as normal, where we get the right-hand side over a common denominator and set the numerators equal to each other. In the two examples above, we would need to solve for 4 variables, and we would have to use the second method.

Suppose now that we have a mix of linear and irreducible quadratic factors. We handle each one separately. For example, we have

$$\frac{x^2}{(x - 1)^2(x^2 + 1)(x - 3)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{Cx + D}{x^2 + 1} + \frac{E}{x - 3}.$$

Again, we would proceed as normal from here: get the right-hand side over a common denominator and set the numerators equal to each other. Lastly, if we have irreducible cubics (i.e. they cannot be factored into quadratic or linear factors), then we have, for example,

$$\frac{x}{(x - 1)(x^3 - x + 17)} = \frac{A}{x - 1} + \frac{Bx^2 + Cx + D}{x^3 - x + 17}.$$

The case for even higher-power factors is similar. We end with some examples of the above concepts showing how to decompose various fractions.

*Example 1.*

$$\frac{1}{x^3 - x} = \frac{1}{x(x - 1)(x + 1)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 1}.$$

*Example 2.*

$$\frac{2x^3 + 3}{x(x - 1)^3} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2} + \frac{D}{(x - 1)^3}.$$

*Example 3.*

$$\frac{-2x}{(x - 2)(x^2 + 5)^2} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 5} + \frac{Dx + E}{(x^2 + 5)^2}.$$

*Example 4.*

$$\frac{1}{(x^3 + x - 7)(x - 1)^2} = \frac{Ax^2 + Bx + C}{x^3 + x - 7} + \frac{D}{x - 1} + \frac{E}{(x - 1)^2}.$$