## Mechanics - Basic Physical Concepts

**Math:** Circle: $2\pi r$, $\pi r^2$; Sphere: $4\pi r^2$, $(4/3)\,\pi r^3$

**Quadratic Eq.:** $a\,x^2 + b\,x + c = 0$, $x = \frac{-b\pm\sqrt{b^2-4\,a\,c}}{2\,a}$

**Cartesian and polar coordinates:**

$$x = r\cos\theta,\ y = r\sin\theta,\quad r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x}$$

**Trigonometry:** $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)$

$$\sin\alpha + \sin\beta = 2\,\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\,\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin 2\,\theta = 2\,\sin\theta\cos\theta, \cos 2\,\theta = \cos^2\,\theta - \sin^2\,\theta$$

$$1-\cos\theta = 2\,\sin^2\frac{\theta}{2},\quad 1+\cos\theta = 2\,\cos^2\frac{\theta}{2}$$

**Vector algebra:** $\vec{A} = (A_x, A_y) = A_x\hat{\imath} + A_y\hat{\jmath}$

**Resultant:** $\vec{R} = \vec{A} + \vec{B} = (A_x + B_x, A_y + B_y)$

**Dot:** $\vec{A}\cdot\vec{B} = A\,B\cos\theta = A_x\,B_x + A_y\,B_y + A_z\,B_z$

**Cross product:** $\hat{\imath}\times\hat{\jmath} = \hat{k},\quad \hat{\jmath}\times\hat{k} = \hat{\imath},\quad \hat{k}\times\hat{\imath} = \hat{\jmath}$

$$\vec{C} = \vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$C = A\,B\sin\theta = A_\perp B = A\,B_\perp$, use right hand rule

**Calculus:** $\frac{d}{dx}x^n = n\,x^{n-1}$, $\frac{d}{dx}\ln x = \frac{1}{x}$,

$\frac{d}{d\theta}\sin\theta = \cos\theta$, $\frac{d}{d\theta}\cos\theta = -\sin\theta$, $\frac{d}{dx}\,\text{const} = 0$

### Measurements

**Dimensional analysis:** *e.g.,*

$F = m\,a \to [M][L][T]^{-2}$, or $F = m\,\frac{v^2}{r} \to [M][L][T]^{-2}$

**Summation:** $\sum_{i=1}^{N}(a\,x_i + b) = a\sum_{i=1}^{N}x_i + b\,N$

### Motion

**One dimensional motion:** $v = \frac{ds}{dt}$, $a = \frac{dv}{dt}$

**Average values:** $\bar{v} = \frac{s_f - s_i}{t_f - t_i}$, $\bar{a} = \frac{v_f - v_i}{t_f - t_i}$

**One dimensional motion (constant acceleration):**

$v(t):\ v = v_0 + a\,t$

$s(t):\ s = \bar{v}\,t = v_0\,t + \frac{1}{2}\,a\,t^2$, $\bar{v} = \frac{v_0+v}{2}$

$v(s):\ v^2 = v_0^2 + 2\,a\,s$

**Nonuniform acceleration:** $x = x_0 + v_0\,t + \frac{1}{2}\,a\,t^2 + \frac{1}{6}\,j\,t^3 + \frac{1}{24}\,s\,t^4 + \frac{1}{120}\,k\,t^5 + \frac{1}{720}\,p\,t^6 + \ldots$, (jerk, snap,...)

**Projectile motion:** $t_{rise} = t_{fall} = \frac{t_{trip}}{2} = \frac{v_{0y}}{g}$

$h = \frac{1}{2}\,g\,t_{fall}^2$, $R = v_{ox}\,t_{trip}$

**Circular:** $a_c = \frac{v^2}{r}$, $v = \frac{2\,\pi\,r}{T}$, $f = \frac{1}{T}$ (Hertz=s$^{-1}$)

**Curvilinear motion:** $a = \sqrt{a_t^2 + a_r^2}$

**Relative velocity:** $\vec{v} = \vec{v}' + \vec{u}$

### Law of Motion and applications

**Force:** $\vec{F} = m\,\vec{a}$, $F_g = m\,g$, $\vec{F}_{12} = -\vec{F}_{21}$

**Circular motion:** $a_c = \frac{v^2}{r}$, $v = \frac{2\,\pi\,r}{T} = 2\,\pi\,r\,f$

**Friction:** $F_{static} \le \mu_s\,N$ $\quad F_{kinetic} = \mu_k\,N$

**Equilibrium (concurrent forces):** $\sum_i \vec{F}_i = 0$

### Energy

**Work (for all F):** $\Delta W = W_{A\to B} = W_B - W_A = F_{\|}\,s = F\,s\cos\theta = \vec{F}\cdot\vec{s} \to \int_A^B \vec{F}\cdot d\vec{s}$ (in Joules)

**Effects due to work done:** $F_{ext} = m\,a + F_c + f_{nc}$

$W_{ext}|_{A\to B} = K_B - K_A + U_B - U_A + W_{diss}|_{A\to B}$

**Kinetic energy:** $K_B - K_A = \int_A^B m\,\vec{a}\cdot d\vec{s}$, $K = \frac{1}{2}\,m\,v^2$

**K (conservative $\vec{F}$):** $U_B - U_A = -\int_A^B \vec{F}\cdot d\vec{s}$

$U_{gravity} = m\,g\,y$, $\quad U_{spring} = \frac{1}{2}\,k\,x^2$

**From $U$ to $\vec{F}$:** $F_x = -\frac{\partial U}{\partial x}$, $F_y = -\frac{\partial U}{\partial y}$, $F_z = -\frac{\partial U}{\partial z}$

$F_{gravity} = -\frac{\partial U}{\partial y} = -m\,g$, $\quad F_{spring} = -\frac{\partial U}{\partial x} = -k\,x$

**Equilibrium:** $\frac{\partial U}{\partial x} = 0$, $\frac{\partial^2 U}{\partial x^2} > 0$ stable, $< 0$ unstable

**Power:** $P = \frac{dW}{dt} = F\,v_{\|} = F\,v\cos\theta = \vec{F}\cdot\vec{v}$ (Watts)

### Collision

**Impulse:** $\vec{I} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i \to \int_{t_i}^{t_f}\vec{F}\,dt$

**Momentum:** $\vec{p} = m\,\vec{v}$

**Two-body:** $x_{cm} = \frac{m_1\,x_1 + m_2\,x_2}{m_1 + m_2}$

$p_{cm} \equiv M\,v_{cm} = p_1 + p_2 = m_1\,v_1 + m_2\,v_2$

$F_{cm} \equiv F_1 + F_2 = m_1\,a_1 + m_2\,a_2 = M\,a_{cm}$

$K_1 + K_2 = K_1^* + K_2^* + K_{cm}$

**Two-body collision:** $\vec{p}_i = \vec{p}_f = (m_1 + m_2)\,\vec{v}_{cm}$

$v_i^* = v_i - v_{cm}$, $\quad v_i' = v_i^{*\prime} + v_{cm}$

**Elastic:** $v_1 - v_2 = -(v_1' - v_2')$,

$v_i^{*\prime} = -v_i^*$, $\quad v_i' = 2\,v_{cm} - v_i$

**Many body center of mass:** $\vec{r}_{cm} = \frac{\sum m_i\,\vec{r}_i}{\sum m_i} = \frac{\int \vec{r}\,dm}{\int m_i}$

**Force on cm:** $\vec{F}_{ext} = \frac{d\vec{p}}{dt} = M\vec{a}_{cm}$, $\quad \vec{p} = \sum\vec{p}_i$

### Rotation of Rigid-Body

**Kinematics:** $\theta = \frac{s}{r}$, $\quad \omega = \frac{v}{r}$, $\quad \alpha = \frac{a_t}{r}$

**Moment of inertia:** $I = \sum m_i\,r_i^2 = \int r^2 dm$

$I_{disk} = \frac{1}{2}\,M\,R^2$, $\quad I_{ring} = \frac{1}{2}\,M\,(R_1^2 + R_2^2)$

$I_{rod} = \frac{1}{12}\,M\,\ell^2$, $\quad I_{rectangle} = \frac{1}{12}\,M\,(a^2 + b^2)$

$I_{sphere} = \frac{2}{5}\,M\,R^2$, $\quad I_{spherical\ shell} = \frac{2}{3}\,M\,R^2$

$I = M\,(\text{Radius of gyration})^2$, $\quad I = I_{cm} + M\,D^2$

**Kinetic energies:** $K_{rot} = \frac{1}{2}\,I\,\omega^2$, $\quad K = K_{rot} + K_{cm}$

**Angular momentum:** $L = r\,m\,v = r\,m\,\omega\,r = I\,\omega$

**Torque:** $\tau = \frac{dL}{dt} = m\,\frac{dv}{dt}\,r = F\,r = I\,\frac{d\omega}{dt} = I\,\alpha$

$W_{ext} = \Delta K + \Delta U + W_f$, $\quad K = K_{rot} + \frac{1}{2}\,m\,v^2$, $\quad P = \tau\,\omega$

### Rolling, angular momentum and torque

**Rolling:** $K = \frac{1}{2}\left(I_c + M\,R^2\right)\omega^2 = \frac{1}{2}\left(\frac{I_c}{R^2} + M\right)v^2$

**Angular momentum:** $\vec{L} = \vec{r}\times\vec{p}$, $L = r_\perp\,p = I\,\omega$

**Torque:** $\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r}\times\frac{d\vec{p}}{dt} = \vec{r}\times\vec{F}$, $\quad \tau = r_\perp\,F = I\,\alpha$

**Gyroscope:** $\omega_p = \frac{d\phi}{dt} = \frac{1}{L}\frac{dL}{dt} = \frac{\tau}{L} = \frac{m\,g\,h}{I\,\omega}$

### Static equilibrium

$\sum\vec{F}_i = 0$, about any point $\sum\vec{\tau}_i = 0$

**Subdivisions:** $\vec{r}_{cm} = \frac{m_A\,\vec{r}_{Acm} + m_B\,\vec{r}_{Bcm}}{m_A + m_B}$

Elastic modulus = stress/strain

**stress:** F/A

**strain:** $\Delta L/L$, $\quad \theta \approx \Delta x/h$, $\quad -\Delta V/V$

## Oscillation motion

$f = \frac{1}{T}, \quad \omega = \frac{2\pi}{T}$

**SHM:** $a = \frac{d^2 x}{dt^2} = -\omega^2 x, \quad \alpha = \frac{d^2\theta}{dt^2} = -\omega^2 \theta$

$x = x_{max} \cos(\omega t + \delta), \quad x_{max} = A$

$v = -v_{max} \sin(\omega t + \delta), \quad v_{max} = \omega A$

$a = -a_{max} \cos(\omega t + \delta) = -\omega^2 x, \quad a_{max} = \omega^2 A$

$E = K + U = K_{max} = \frac{1}{2} m (\omega A)^2 = U_{max} = \frac{1}{2} k A^2$

*Spring:* $m a = -k x$

*Simple pendulum:* $m a_\theta = m \alpha \ell = -m g \sin\theta$

*Physical pendulum:* $\tau = I \alpha = -m g d \sin\theta$

*Torsion pendulum:* $\tau = I \alpha = -\kappa \theta$

## Gravity

$\vec{F}_{21} = -G \frac{m_1 m_2}{r_{12}^2} \hat{r}_{12}, \quad \text{for } r \geq R, \quad g(r) = G \frac{M}{r^2}$

$G = 6.67259 \times 10^{-11}\ \mathrm{N\,m^2/kg^2}$

$R_{earth} = 6370$ km, $\quad M_{earth} = 5.98 \times 10^{24}$ kg

**Circular orbit:** $a_c = \frac{v^2}{r} = \omega^2 r = \left(\frac{2\pi}{T}\right)^2 r = g(r)$

$U = -G \frac{m M}{r}, \quad E = U + K = -\frac{G m M}{2r}$

$F = -\frac{dU}{dr} = -m G \frac{M}{r^2} = -m \frac{v^2}{r}$

**Kepler's Laws of planetary motion:**

*i)* elliptical orbit, $r = \frac{r_0}{1 - \epsilon \cos\theta} \quad r_1 = \frac{r_0}{1+\epsilon}, \quad r_2 = \frac{r_0}{1-\epsilon}$

*ii)* $L = r m \frac{\Delta r_\perp}{\Delta t} \longrightarrow \frac{\Delta A}{\Delta t} = \frac{1}{2} \frac{r \Delta r_\perp}{\Delta t} = \frac{L}{2m} = \text{const.}$

*iii)* $G \frac{M}{a^2} = \left(\frac{2\pi a}{T}\right)^2 \frac{1}{a}, \quad a = \frac{r_1 + r_2}{2}, \quad T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$

**Escape kinetic energy:** $E = K + U(R) = 0$

## Fluid mechanics

**Pascal:** $P = \frac{F_{\perp 1}}{A_1} = \frac{F_{\perp 2}}{A_2}, \quad 1\ \text{atm} = 1.013 \times 10^5\ \mathrm{N/m^2}$

**Archimedes:** $B = M g$, $\quad$ Pascal=$\mathrm{N/m^2}$

$P = P_{atm} + \rho g h$, $\quad$ with $P = \frac{F_\perp}{A}$ and $\rho = \frac{m}{V}$

$F = \int P\, dA \longrightarrow \rho g \ell \int_0^h (h - y)\, dy$

**Continuity equation:** $A v = \text{constant}$

**Bernoulli:** $P + \frac{1}{2} \rho v^2 + \rho g y = \text{const}, \qquad P \geq 0$

## Wave motion

**Traveling waves:** $y = f(x - v t), \quad y = f(x + v t)$

**In the positive $x$ direction:** $y = A \sin(k x - \omega t - \phi)$

$T = \frac{1}{f}, \quad \omega = \frac{2\pi}{T}, \quad k = \frac{2\pi}{\lambda}, \quad v = \frac{\omega}{k} = \frac{\lambda}{T}$

**Along a string:** $v = \sqrt{\frac{F}{\mu}}$

**General:** $\Delta E = \Delta K + \Delta U = \Delta K_{max}$

$P = \frac{\Delta E}{\Delta t} = \frac{1}{2} \frac{\Delta m}{\Delta t} (\omega A)^2$

**Waves:** $\frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta x} \cdot \frac{\Delta x}{\Delta t} = \frac{\Delta m}{\Delta x} \cdot v$

$P = \frac{1}{2} \mu v (\omega A)^2, \quad$ with $\mu = \frac{\Delta m}{\Delta x}$

**Circular:** $\frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta A} \cdot \frac{\Delta A}{\Delta r} \cdot \frac{\Delta r}{dt} = \frac{\Delta m}{\Delta A} \cdot 2\pi r v$

**Spherical:** $\frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta V} \cdot 4\pi r^2 v$

## Sound

$v = \sqrt{\frac{B}{\rho}}, \quad s = s_{max} \cos(k x - \omega t - \phi)$

$\Delta P = -B \frac{\Delta V}{V} = -B \frac{\partial s}{\partial x}$

$\Delta P_{max} = B \kappa s_{max} = \rho v \omega s_{max}$

**Piston:** $\frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta V} \cdot \frac{A \Delta x}{\Delta t} = \rho A v$

**Intensity:** $I = \frac{P}{A} = \frac{1}{2} \rho v (\omega s_{max})^2$

**Intensity level:** $\beta = 10 \log_{10} \frac{I}{I_0}, \quad I_0 = 10^{-12}\ \mathrm{W/m^2}$

**Plane waves:** $\psi(x, t) = c \sin(k x - \omega t)$

**Circular waves:** $\psi(r, t) = \frac{c}{\sqrt{r}} \sin(k r - \omega t)$

**Spherical:** $\psi(r, t) = \frac{c}{r} \sin(k r - \omega t)$

**Doppler effect:** $\lambda = v T, \quad f_0 = \frac{1}{T}, \quad f' = \frac{v'}{\lambda'}$

Here $v' = v_{sound} \pm v_{observer}$, is wave speed relative to moving observer and $\lambda' = (v_{sound} \pm v_{source})/f_0$, detected wave length established by moving source of frequency $f_0$. $f_{received} = f_{reflected}$

**Shock waves:** Mach Number= $\frac{v_{source}}{v_{sound}} = \frac{1}{\sin\theta}$

## Superposition of waves

**Phase difference:** $\sin(k x - \omega t) + \sin(k x - \omega t - \phi)$

**Standing waves:** $\sin(k x - \omega t) + \sin(k x + \omega t)$

**Beats:** $\sin(kx - \omega_1 t) + \sin(k x - \omega_2 t)$

**Others:** $a \cos(k x - \omega t) + b \sin(k x - \omega t)$

$y = \sin(k x - \omega t), \ z = \sin(k x - \omega t)$

**Fundamental modes:** Sketch wave patterns

*String:* $\frac{\lambda}{2} = \ell$, *Rod clamped middle:* $\frac{\lambda}{2} = \ell$,

*Open-open pipe:* $\frac{\lambda}{2} = \ell$, *Open-closed pipe:* $\frac{\lambda}{4} = \ell$

## Temperature and heat

**Conversions:** $F = \frac{9}{5} C + 32°, \qquad K = C + 273.15°$

**Constant volume gas thermometer:** $T = a P + b$

**Thermal expansion:** $\alpha = \frac{1}{\ell} \frac{d\ell}{dT}, \quad \beta = \frac{1}{V} \frac{dV}{dT}$

$\Delta\ell = \alpha \ell \Delta T, \quad \Delta A = 2 \alpha A \Delta T, \quad \Delta V = 3 \alpha V \Delta T$

**Ideal gas law:** $\qquad P V = n R T = N k T$

$R = 8.314510$ J/mol/K $= 0.0821$ L atm/mol/K

$k = 1.38 \times 10^{-23}$J/K, $\quad N_A = 6.02 \times 10^{23}$, 1 cal=4.19 J

**Calorimetry:** $\Delta Q = c m \Delta T, \ \Delta Q = L \Delta m$

**First law:** $\Delta U = \Delta Q - \Delta W, \quad W = \int P\, dV$

**Conduction:** $H = \frac{\Delta Q}{\Delta t} = -k A \frac{\Delta T}{\Delta \ell}, \quad \Delta T_i = \frac{-H}{A} \frac{\ell_i}{k_i}$

**Stefan's law:** $P = \sigma A e T^4, \quad \sigma = 5.67 \times 10^{-8} \frac{W}{m^2 K^4}$

## Kinetic theory of gas

**Ideal gas:** $\Delta p_x = 2 m v_x, \quad F = \frac{\Delta p_x}{\Delta t} = \frac{m v_x^2}{d}$

**Pressure:** $P = \frac{N \overline{F}}{A} = \frac{m N}{V} \overline{v_x^2} = \frac{m N}{3V} \overline{v^2}$

$P = \frac{2}{3} \frac{N}{V} \overline{K}, \quad \overline{K}_x = \frac{\overline{K}}{3} = \frac{1}{2} k T, \quad T = 273 + T_c$,

$P V = N k T, \quad n = N/N_A, \quad k = 1.38 \times 10^{-23}$ J/K,

$N_A = 6.02214199 \times 10^{23}$ #/kg/mole

**Constant V:** $\qquad \Delta Q = \Delta U = n C_V \Delta T$

**Constant P:** $\qquad \Delta Q = n C_P \Delta T$

$\gamma = \frac{C_P}{C_V}, \quad C_P - C_V = R$

$C_V = \frac{d}{2} R$, $\quad$ for transl.+rot+vib, $\quad d = 3 + 2 + 2$

**Adiabatic expansion:** $P V^\gamma = \text{constant}$

**Mean free path:** $\ell = \frac{v_{rms} t}{(v_{rel})_{rms} t} \frac{1}{\pi d^2 n_V} = \frac{1}{\sqrt{2} \pi d^2 n_V}$