

## Summary

A **power series representation** of a function  $f(x)$  is of the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n,$$

where each  $c_n$  is a constant. The most important power series representation to remember is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{where } |x| < 1.$$

We can find a power series representation for many rational functions by rewriting them in the form  $\frac{1}{1-x}$ . Example 1 covers this. We can find power series representations for many more functions if we can find power series representations for  $\int f(x) dx$  or  $f'(x)$ . Examples 2 and 3 cover these cases respectively.

If we find a power series representation for  $\int f(x) dx$  we can take the derivative of that to get a representation for  $f(x)$ . The index for the series will go up by 1 each time we take a derivative. When we find a power series representation for  $f'(x)$ , we can integrate that to get a representation for  $f(x)$ . To compute the resulting constant of integration, we plug in  $x = 0$ .

## What is a Power Series Representation?

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$

where each  $c_n$  is a constant. Many functions, such as  $\frac{2x}{(1+x)^3}$  and  $\tan^{-1}(2x)$ , can be represented as power series in this way. We first start with such a representation, called a **power series representation**, for the function  $\frac{1}{1-x}$ .

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{where } |x| < 1. \quad (\star)$$

This equation comes from the formula for a geometric series:

$$\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}, \quad \text{where } |r| < 1,$$

where we replace  $a$  with 1 and  $r$  with  $x$ . The formula  $(\star)$  is of the general form described above, with  $c_n = 1$  for all  $n$ . We will use this equality to determine power series representations for many other functions.

## Power Series Representations of Rational Functions

Consider the function  $\frac{1}{1-x^2}$ . This looks a lot like  $\frac{1}{1-x}$ , but with  $x^2$  instead of  $x$ . Looking at  $(\star)$ , we plug  $x^2$  into  $x$  to get

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}, \quad \text{where } |x^2| < 1.$$

Similarly, for  $\frac{1}{1-2x}$ , we can plug in  $2x$  for  $x$  to get

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n, \quad \text{where } |2x| < 1.$$

# Power Series Representations of Functions

Here, note that  $|2x| < 1$  implies  $-\frac{1}{2} < x < \frac{1}{2}$ . We will often omit this condition, but it is always assumed. We now turn our attention to the function  $\frac{2x}{1-x}$ . We write this as  $2x \left( \frac{1}{1-x} \right)$ , which yields

$$\frac{2x}{1-x} = 2x \left( \sum_{n=0}^{\infty} x^n \right) = \sum_{n=0}^{\infty} (2x)x^n = \sum_{n=0}^{\infty} 2x^{n+1}.$$

In this example, we factored out the numerator to get a fraction of the form  $(\star)$ . Now consider  $\frac{1}{1+x}$ . We cannot yet use  $(\star)$ , because there is a “+” in the denominator rather than a “-”. To get the minus sign, we use a double negative:

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Lastly, consider the function  $\frac{1}{2-x}$ . Again, this is close to  $\frac{1}{1-x}$ , but we have a 2 in the denominator instead of a 1. We rewrite  $2-x$  as  $\frac{1}{2}(1-2x)$  to get

$$\frac{1}{2-x} = \frac{1}{\frac{1}{2}(1-2x)} = 2 \left( \frac{1}{1-2x} \right) = 2 \left( \sum_{n=0}^{\infty} (2x)^n \right) = \sum_{n=0}^{\infty} 2^{n+1} x^n.$$

To sum up, we want to get our rational functions of the form  $\frac{1}{1-f(x)}$ , where  $f(x)$  is some function of  $x$ , such as  $x^2$  or  $-x$ . The power series representation is valid on the interval  $-1 < f(x) < 1$ . Here is an example that brings all of this together:

*Example 1.* Find a power series representation for the function  $\frac{3x^2}{4+2x^3}$ .

First, we get a 1 for the constant term in the denominator:

$$\frac{3x^2}{4+2x^3} = \frac{3x^2}{4 \left( 1 + \frac{1}{2}x^3 \right)} = \frac{\frac{3}{4}x^2}{1 + \frac{1}{2}x^3}.$$

Next, we use a double negative to get a minus sign in the denominator:

$$\frac{\frac{3}{4}x^2}{1 + \frac{1}{2}x^3} = \frac{\frac{3}{4}x^2}{1 - \left( -\frac{1}{2}x^3 \right)}.$$

Lastly, we pull out the numerator and then use our series representation, with  $-1 < -\frac{1}{2}x^3 < 1$ :

$$\frac{\frac{3}{4}x^2}{1 - \left( -\frac{1}{2}x^3 \right)} = \frac{3}{4}x^2 \left( \sum_{n=0}^{\infty} \left( -\frac{1}{2}x^3 \right)^n \right) = \sum_{n=0}^{\infty} \left( \frac{3}{4}x^2 \right) \frac{(-1)^n x^{3n}}{2^n}.$$

Simplifying this expression gives us a final answer of

$$\frac{3x^2}{4+2x^3} = \sum_{n=0}^{\infty} \frac{3(-1)^n x^{3n+2}}{2^{n+2}}.$$

## Differentiation, Integration, and Power Series Representations

So far we have seen how to get power series representations of certain rational functions, but what if we have a function like, say,  $\frac{1}{1-2x+x^2}$ ? It's not clear how to get this function in the form  $\frac{1}{1-x}$ . If, however, we write this function as  $\frac{1}{(1-x)^2}$ , we can then integrate, getting

$$\int \frac{1}{(1-x)^2} dx = \frac{-1}{1-x} + C.$$

# Power Series Representations of Functions

Using  $(\star)$ , this yields

$$\int \frac{1}{(1-x)^2} dx = \frac{-1}{1-x} + C = C - \sum_{n=0}^{\infty} x^n.$$

To “undo” the integral around  $\frac{1}{(1-x)^2}$ , we could take the derivative of both sides. Let’s see what this does to the right-hand side:

$$\frac{d}{dx} \left[ C - \sum_{n=0}^{\infty} x^n \right] = - \sum_{n=1}^{\infty} nx^{n-1}.$$

To take the derivative of the series we essentially took the derivative of the summand (the  $x^n$ ). To verify this step, we can write out the first few terms of the series:

$$\frac{d}{dx} \left[ \sum_{n=0}^{\infty} x^n \right] = \frac{d}{dx} [1 + x + x^2 + x^3 + \dots] = 1 + 2x + 3x^2 + \dots = \sum_{n=1}^{\infty} nx^{n-1}.$$

Note that our index for this series is 1 rather than 0. This happens because the constant term in  $\sum_{n=0}^{\infty} x^n$ , namely the first term, 1, goes to 0 when we take the derivative, so we shift over from  $n = 0$  to  $n = 1$ . This indexing change often confuses students, so it may be helpful to write out the terms of the series as we did above to check that the indexing is correct.

We now have

$$\frac{1}{(1-x)^2} = \frac{d}{dx} \left[ C - \sum_{n=0}^{\infty} x^n \right] = - \sum_{n=1}^{\infty} nx^{n-1},$$

which is our desired power series representation. Now let’s consider the function  $\tan^{-1}(2x)$ . This is not a rational function, so it’s not quite clear where to start. If we recall, however, that the derivative of  $\tan^{-1}$  is  $\frac{1}{1+x^2}$ , this suggests that we can take a derivative to get a rational function:

$$\frac{d}{dx} [\tan^{-1}(2x)] = \frac{2}{1+4x^2} = 2 \sum_{n=0}^{\infty} (-4x^2)^n = \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} x^{2n}.$$

This power series representation comes from methods we covered in the previous section, rewriting  $\frac{2}{1+4x^2}$  as  $2 \left( \frac{1}{1-(-4x^2)} \right)$ . In the last example, we found that the integral of our function had a series representation, so we took the derivative of that series representation to get back to our original function. Now we have found a series representation for the *derivative* of our function, so we will take an *integral* to get back to our original function. This gives

$$\tan^{-1}(2x) = \int \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} x^{2n} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}.$$

We will figure out what  $C$  is soon. Note that when taking our integral, we treat all terms without an  $x$  in them as constants, even if they have an “ $n$ ” in them. It’s a common mistake for students to integrate  $(2x)^n$  (using the Power Rule) as  $((2x)^{n+1})/(n+1)$ , which equals  $(2^{n+1}x^{n+1})/(n+1)$ . In fact, if we write  $(2x)^n$  as  $2^n x^n$ , we see that the integral is actually  $(2^n x^{n+1})/(n+1)$  because  $2^n$  is a constant with respect to  $x$ .

It may be helpful, as before, to write out the first few terms when taking the integral:

$$\begin{aligned} \int \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} x^{2n} dx &= \int 2 - 8x^2 + 32x^4 - 128x^6 + \dots dx \\ &= C + \left( 2x - \frac{8x^3}{3} + \frac{32x^5}{5} - \frac{128x^7}{7} + \dots \right) \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}. \end{aligned}$$

# Power Series Representations of Functions

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Now we turn our attention to  $C$ . If we plug in 0 for  $x$ , the series vanishes since every term is multiplied by  $x$  to some positive power. Furthermore, we want this expression to be equal to  $\tan^{-1}(2x)$ , so plugging in 0 for  $x$  gives

$$\tan^{-1}(2(0)) = \tan^{-1}(0) = 0.$$

This means that  $C = 0$ , and we have

$$\tan^{-1}(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}.$$

*Example 2.* Find a power series representation for  $\frac{2x}{(1+x)^3}$ .

We will find a power series representation for  $\frac{2}{(1+x)^3}$  and then multiply that by  $x$ . First note that  $\frac{2}{(1+x)^3}$  is the second derivative of  $\frac{1}{1+x}$ , which has power series  $\sum_{n=0}^{\infty} (-1)^n x^n$ . Thus, we have

$$\frac{2}{(1+x)^3} = \frac{d^2}{dx^2} \left[ \frac{1}{1+x} \right] = \frac{d^2}{dx^2} \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] = \sum_{n=2}^{\infty} (-1)^n n(n+1) x^{n-2}.$$

To see how we get the right-hand side, consider that

$$\frac{d^2}{dx^2} \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] = \frac{d}{dx} \left[ \sum_{n=1}^{\infty} (-1)^n n x^{n-1} \right],$$

as we saw before. When we take the derivative of this, we just increase the index by 1 again. If this is still not clear, try writing out the first several terms of  $\sum_{n=0}^{\infty} (-1)^n x^n$  and taking the second derivative. Recall that we factored out an  $x$  in the beginning. We now put that back in to get a final answer of

$$\frac{2x}{(1+x)^3} = x \left( \frac{2}{(1+x)^3} \right) = x \left( \sum_{n=2}^{\infty} (-1)^n n(n+1) x^{n-2} \right) = \sum_{n=2}^{\infty} (-1)^n n(n+1) x^{n-1}.$$

*Example 3.* Find a power series representation for  $\ln(5+x^3)$ .

The derivative of  $\ln(5+x^3)$  is  $\frac{3x^2}{5+x^3}$ . This has a power series representation of

$$\frac{3x^2}{5+x^3} = 3x^2 \left( \frac{\frac{1}{5}}{1 - (-\frac{1}{5}x^3)} \right) = \sum_{n=0}^{\infty} \frac{3(-1)^n x^{3n+2}}{5^{n+1}}.$$

We must now integrate this to get back to  $\ln(5+x)$ . This gets

$$\ln(5+x^3) = \int \sum_{n=0}^{\infty} \frac{3(-1)^n x^{3n+2}}{5^{n+1}} dx = C + \sum_{n=0}^{\infty} \frac{3(-1)^n x^{3n+3}}{5^{n+1}(3n+3)}.$$

To find  $C$ , we plug in  $x = 0$  to get  $\ln(5+0^3) = \ln 5$  on the left-hand side of this equation. As for the right-hand side, the fact that  $x = 0$  makes every term in the summation equal to 0. Thus, the right-hand side simply equals  $C$ , giving us  $C = \ln 5$ . This gives us our final answer:

$$\ln(5+x^3) = \ln 5 + \sum_{n=0}^{\infty} \frac{3(-1)^n x^{3n+3}}{5^{n+1}(3n+3)}.$$