**Related Rates Problems**

   Consider a ladder propped up against a wall. Given the speed at which the end on the floor is moving, how quickly is the end on the wall moving? Now think of a launched rocket flying through the air. If, at a given point in time, Sally observe it from 100 meters away moving at a certain speed, how quickly is it moving away from her? Lastly, if a water ballloon is filling up at a certain rate, how quickly does its surface area change?

   All of these problems are examples of **Related Rates** problems, in that the rates of interest are related to rates given in the problem. In this handout, we'll discuss each of the three examples above. First, we'll state the general process that we'll use to tackle each problem:

1. Assign variables to the relevant quantities in the problem. It often helps to draw a picture as well.

2. Find a relation between the given rate and the rate of interest wish to find.

3. Use this relation to solve for the desired rate.

**The Ladder Example**

   Consider a 10-foot ladder propped up against the wall, such that one end is 8 feet up the wall and its other end is 6 feet away from the wall. If you push the base of the ladder towards the wall at a rate of 3 feet per second, how quickly is the end on the wall rising? We will use the process described above to answer this question.

1. First, we'll assign variables. Let $x$ be the distance from the base of the wall to the end of the ladder that's on the floor, and let $y$ be the distance from the base of the wall to the end of the ladder touching the wall. Then $x = 6$ and $y = 8$. Both $x$ and $y$ are changing in time. We are given the rate $\frac{dx}{dt} = -3$, and we wish to find $\frac{dy}{dt}$. For the second step, it will be helpful to draw all this information in a picture:
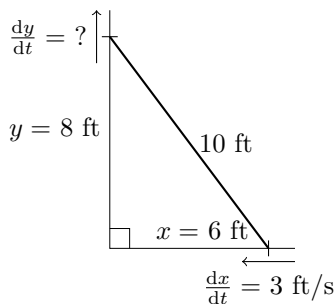


   Note that, while we list $\frac{dx}{dt}$ as a positive number in the picture, the arrow is pointing to the left, so the distance $x$ is actually decreasing.

2. Next, we must find a relation between the given rate, $\frac{dx}{dt}$, and the desired rate, $\frac{dy}{dt}$. To this end, we'll find a relationship between $x$ and $y$ and differentiate both sides with respect to $t$. The picture immediately suggests the relation: the Pythagorean Theorem gives

$$x^2 + y^2 = 10^2 = 100.$$

3. Lastly, we can solve for the rate. Taking the derivative (with respect to $t$) of both sides of the above equation gives

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$2y\frac{dy}{dt} = -2x\frac{dx}{dt}$$
$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

Plugging in information from the problem, this gives

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{(6)}{(8)}(-3)\\
&= -\frac{-9}{4}\\
&= \frac{9}{4} = 2.25 \text{ feet per second.}
\end{aligned}$$

**The Rocket Example**

Sally watches a rocket launch from 100 meters away. When the rocket is 75 meters in the air, it is travelling upwards at a speed of 200 meters per second. What is the speed at which the distance between Sally and the rocket is increasing?

1. We first assign variables. Let $h$ be the height of the rocket in the air, and let $D$ be the distance between the observer and the rocket. Then $h = 75$, and $\frac{\mathrm{d}h}{\mathrm{d}t} = 200$, and we wish to find $\frac{\mathrm{d}D}{\mathrm{d}t}$. As before, we draw this in a picture:
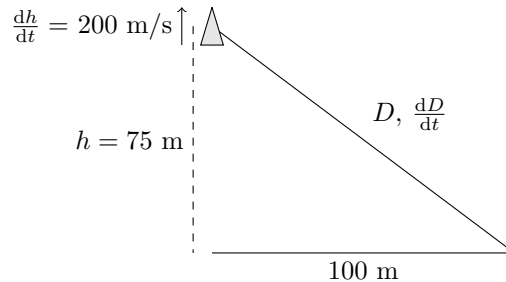


2. As before, the picture suggests the relation between the variables. The Pythagorean Theorem gives

$$D^2 = h^2 + 100^2 = h^2 + 10,000.$$

3. We solve for the rate by taking the derivative of both sides with respect to $t$:

$$\begin{aligned}
2D\frac{\mathrm{d}D}{\mathrm{d}t} &= 2h\frac{\mathrm{d}h}{\mathrm{d}t}\\
\frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{h}{D}\frac{\mathrm{d}h}{\mathrm{d}t}.
\end{aligned}$$

We want to now plug in for the variables on the right-hand side, but we don't know $D$. The Pythagorean Theorem gives $D = 125$, and with this in hand we get our final answer:

$$\begin{aligned}
\frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{(75)}{(125)}(200)\\
&= \frac{3}{5}(200)\\
&= 120 \text{ meters per second.}
\end{aligned}$$

**Water Balloon Example**

Suppose a spherical water balloon is filling up at a rate of $8\pi$ cubic centimeters per second. When the water balloon has a volume of $36\pi$ cubic centimeters, how quickly does the surface area of the water balloon increase?

1. Let $V$ be the volume of the water balloon. We are given $\frac{\mathrm{d}V}{\mathrm{d}t} = 8\pi$, and $V = 36\pi$. We wish to find the rate of change of the surface area, which we'll label $S$. In this problem, it may not be as helpful to draw a picture as in the other examples. The situation in the problem implies that we should use one variable which is not explicitly given: the radius of the water balloon. Let's call this $r$. This will be the key to relating the volume and the surface area.

2. The volume and surface area of the water ballloon are given by the respective formulas

$$V = \frac{4}{3}\pi r^3, \quad \text{and} \quad S = 4\pi r^2.$$

We could manipulate these two equations to write $V$ in terms of $S$ and then proceed as usual from there. The relation would be

$$V = \frac{4}{3}\pi \left( \sqrt{\frac{S}{4\pi}} \right)^3,$$

which we could simplify further if we wanted. It would be a bit messy to take the derivative of this, however, and we now give another method.

3. Differentiate both sides of the volume equation and both sides of the surface area equation with respect to $t$, to get

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t}, \quad \text{and} \quad \frac{\mathrm{d}S}{\mathrm{d}t} = 8\pi r \frac{\mathrm{d}r}{\mathrm{d}t}.$$

Now, we will use the first equation to get $r$ and $\frac{\mathrm{d}r}{\mathrm{d}t}$ and plug these into the second equation. Firstly, we use that $V = 27\pi$ to get

$$36\pi = \frac{4}{3}\pi r^3$$
$$27 = r^3$$
$$r = 3 \text{ centimeters.}$$

Next, we plug this value as well as $\frac{\mathrm{d}V}{\mathrm{d}t} = 8\pi$ into the first of the two equations above:

$$8\pi = 4\pi(3)^2 \frac{\mathrm{d}r}{\mathrm{d}t}$$
$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{2}{9} \text{ centimeters per second.}$$

With both $r$ and $\frac{\mathrm{d}r}{\mathrm{d}t}$ in hand, we use the second equation to get our final answer:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 8\pi(3)\left(\frac{2}{9}\right)$$
$$= \frac{16}{3}\pi \text{ square centimeters per second.}$$