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Solving Linear Systems of Equations: Row-reducing a Matrix

Linear Systems of Equations and Matrices

A **linear system of equations**, also known as a **system of linear equations**, is a set of some number polynomial equations in some number of variables where the highest degree is 1. Note that this excludes cross terms. Examples of linear equations include

$$\begin{aligned} x + y + z &= 1 \\ 2x - y &= 3 \\ 3z - y &= 5\,, \end{aligned}$$

and examples of non-linear equations include

$$\begin{aligned} x + y + \cos z &= 1 && \text{(not a polynomial)} \\ x^2 - y + z &= 3 && (x \text{ has degree greater than } 1) \\ xy - y + z &= 2 && (\text{the } xy \text{ is a crossterm}). \end{aligned}$$

A **solution** to a system of equations is a set of values such that all the equations are simultaneously satisfied. The first three examples we gave of linear equations form a linear system, and a solution to this system of equations would be a set of (x, y, z) values such that all of the equations are simultaneously satisfied.

One use of a **matrix** is as a succinct way of writing a system of linear equations. Here is a system of 2 equations in 3 unknowns:

$$\begin{aligned} x + 2y - \ z &= 2 \\ 4x + \ y - 3z &= -1\,. \end{aligned}$$

We'll give our matrix and then explain it. The matrix is as follows:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 4 & 1 & -3 & -1 \end{array}\right).$$

The first column gives the coefficients of the x-terms in the two equations, the second column gives the coefficients of the y-terms, and the third column gives the coefficients of the z-terms. The last column represents the right-hand side of our equations, and is separated from the coefficients by a vertical bar. This type of matrix is actually called an **augmented matrix**. Sometimes, we omit this last column, which gives the **coefficient matrix** (or just "matrix").

Let's convert one more system of equations into an augmented matrix:

$$\begin{aligned} 2x - 4y - 6z &= -10 \\ 3x + 2y - \ z &= 1 \\ -x + 6y - 2z &= -14\,. \end{aligned}$$

This gives us

$$\left(\begin{array}{ccc|c} 2 & -4 & -6 & -10 \\ 3 & 2 & -1 & 1 \\ -1 & 6 & -2 & -14 \end{array}\right).$$

Row Operations of a Matrix

From elementary algebra we know that we can add and subtract equations to get new equations and that this does not change the solution. For example, let's solve the linear system

$$\begin{aligned} 2x - 4y &= 2 \\ x + \ y &= 4\,. \end{aligned}$$

To solve this, we could first multiply both sides of the second equation by 2 and then subtract the first equation from the second. Then, we could divide both sides of this new equation by 6 to get $y = 1$. Plugging $y = 1$ into the first equation would get $2x = 6$, and dividing both sides of this by 2 gets $x = 3$. This process, step-by-step, gives:

$$\begin{aligned} 2x - 4y &= 2 \\ x + y &= 4 \end{aligned} \xrightarrow{(1)} \begin{aligned} 2x - 4y &= 2 \\ 2x + 2y &= 8 \end{aligned} \xrightarrow{(2)} \begin{aligned} 2x - 4y &= 2 \\ 6y &= 6 \end{aligned} \xrightarrow{(3)}$$

$$\begin{aligned} 2x - 4y &= 2 \\ y &= 1 \end{aligned} \xrightarrow{(4)} \begin{aligned} 2x &= 6 \\ y &= 1 \end{aligned} \xrightarrow{(5)} \begin{aligned} x &= 3 \\ y &= 1\,. \end{aligned}$$

Let's write the original system of equations as a matrix:

$$\left(\begin{array}{cc|c} 2 & -4 & 2 \\ 1 & 1 & 4 \end{array}\right).$$

In step (1), we multiplied both sides of the second equation by 2. This corresponds to multiplying all entries of Row 2 by 2. In general, we can scale a row by any constant we wish without changing the solutions of our system just as we can multiply both sides of an equation by any constant we wish.

In step (2), we subtracted the first equation from the second to get $6y = 6$. This corresponds to subtracting Row 1 from Row 2. Note that we change Row 2 and leave Row 1 fixed. Just as we can add and subtract equations, we can add and subtract rows; however, we must always be mindful of which row we're altering. It would be incorrect to change Row 1 when subtracting Row 1 from Row 2. As a rule of thumb, the operation "Row x − Row y", represents altering Row x, the Row to the left of the minus sign. When adding two rows together we can change either row, but it is best practice to write the operation as "Row x + Row y" if you are changing Row x, not Row y.

One other step we could do is switch the order of the equations. We could, for example, list the second equation first and the first equation second. This corresponds to swapping the rows of a matrix. This may not seem very useful in terms of equations, but in terms of matrices it will be helpful.

When manipulating matrices in this way, the easiest way to avoid mistakes it to move one step at a time. While it is permissible to add a Row to more than one other Row at the same time, and while it is permissible to scale more than one row at once (and even by different constants), it is safest to avoid doing so until one has more experience with row reduction.

To end this section, let's translate our work from before into row operations on matrices:

$$\left(\begin{array}{cc|c} 2 & -4 & 2 \\ 1 & 1 & 4 \end{array}\right) \quad R_2 \overset{(1)}{\to} 2R_2 \quad \left(\begin{array}{cc|c} 2 & -4 & 2 \\ 2 & 2 & 8 \end{array}\right) \quad R_2 \overset{(2)}{\to} R_2 - R_1$$

$$\left(\begin{array}{cc|c} 2 & -4 & 2 \\ 0 & 6 & 6 \end{array}\right) \quad R_2 \overset{(3)}{\to} \frac{1}{6}R_2 \quad \left(\begin{array}{cc|c} 2 & -4 & 2 \\ 0 & 1 & 1 \end{array}\right) \quad R_1 \overset{(4)}{\to} R_1 + 4R_2$$

$$\left(\begin{array}{cc|c} 2 & 0 & 6 \\ 0 & 1 & 1 \end{array}\right) \quad R_1 \overset{(5)}{\to} \frac{1}{2}R_1 \quad \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right).$$

Step (4) in this method is a bit different from the step (4) when we were simply looking at the equations. Instead of plugging in $y = 1$, we turn the -4 in the first row into a 0.

Notice how the left-hand side having 1's on the diagonal and 0's elsewhere in the final matrix allows us to easily read off the x- and y-values since such a matrix corresponds to the equations $x = 3$ and $y = 1$. For every matrix, there is a systematic way to approach this goal of 1's on the diagonal and 0's elsewhere, though we cannot always completely achieve this. This method is called **row-reduction**.

The Method of Matrix Row-reduction

Recall that we have three ways to manipulate matrices:

1. We can scale all the entries of a row by a constant.

2. We can add and subtract rows entry-by-entry. When subtracting Row y from Row x, we change Row x and leave Row y unchanged.

3. We can swap the rows of a matrix.

The general goal of row-reduction is to get 1's on the diagonal entries and 0's elsewhere on the left-hand part of our matrix. The resulting matrix translates to the simplest system of linear equations. Often we cannot quite achieve this, but we can get as close to this as possible for any matrix using the process of row-reduction.

In row-reduction, we focus on the part of the matrix to the left of the vertical bar. While doing our row operations, we must remember to change the entries to the right of the vertical bar accordingly, but otherwise the right side of the vertical bar doesn't affect whether the matrix is row-reduced or not. The matrix-form we desire, called **reduced row echelon form**, has the following properties:

- Any rows with all zeros are below any nonzero rows. In the example above, there were no rows with all zeros.

- The leading coefficient of a row, called the **pivot**, is strictly to the right of the pivot of the row above it. In the example above, the matrix after step 4 had pivots 2 (first row) and 1 (second row).

- Every pivot is 1, and each pivot is the only nonzero entry in its respective column. In step 5 in the example above we changed the pivot of the first row from 1 to 2. In step 3, we eliminated the nonzero coefficient, -4, above the pivot in the second row.

To achieve reduced row echelon form, we follow an algorithm:

1. Get a pivot of 1 in the first row. We want the upper-left entry to be 1. We can swap the first row with a row that has a 1 in the first column, or we can scale the first row.

2. Get a 0 in the first column for every row below the first. We can do this by subtracting a suitably scaled first row from each of the following rows. For example, if the second row has a 3 in the first column, we would do $R_2 \to R_2 - 3R_1$.

3. Get a pivot in the second row as in the first step. Then, just like in the second step, make all the entries in the second column below the second row equal to 0.

4. Repeat step 3 for each of the columns until the last nonzero row has only one nonzero entry, which will be in the rightmost column (still to the left of the veritcal bar).

5. Make the pivot of the last nonzero row equal to 1.

6. Use the last nonzero row to get a 0 in the last column for every other row.

7. In the next row up, get a 0 for every other row in the next-to-last column.

8. Repeat until achieving reduced row echelon form.

To illustrate this process, we will solve the system

$$\begin{array}{l} 2x - 4y - 6z = -10 \\ 3x + 2y - z = 1 \\ -x + 6y - 2z = -14 \end{array} \quad \longrightarrow \quad \left(\begin{array}{ccc|c} 2 & -4 & -6 & -10 \\ 3 & 2 & -1 & 1 \\ -1 & 6 & -2 & -14 \end{array}\right).$$

For step 1, we scale R_1 by $\frac{1}{2}$ (note that we could have also swapped R_1 with R_3 instead and followed a different path to the same end):

$$\left(\begin{array}{ccc|c} 2 & -4 & -6 & -10 \\ 3 & 2 & -1 & 1 \\ -1 & 6 & -2 & -14 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{ccc|c} 1 & -2 & -3 & -5 \\ 3 & 2 & -1 & 1 \\ -1 & 6 & -2 & -14 \end{array}\right).$$

Next, for step 2 we subtract $3R_1$ from R_2 and then add R_1 to R_3:

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & -5 \\ 3 & 2 & -1 & 1 \\ -1 & 6 & -2 & -14 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & -3 & -5 \\ 0 & 8 & 8 & 16 \\ -1 & 6 & -2 & -14 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & -5 \\ 0 & 8 & 8 & 16 \\ 0 & 4 & -5 & -19 \end{array}\right).$$

For the third step, we want to get our pivot in the second row, and then make the entry below it equal to 0. Let's first scale R_2 by $\frac{1}{8}$. Then we'll do $R_3 \to R_3 - 4R_2$:

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & -5 \\ 0 & 8 & 8 & 16 \\ 0 & 4 & -5 & -19 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & -3 & -5 \\ 0 & 1 & 1 & 2 \\ 0 & 4 & -5 & -19 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & -5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -9 & -27 \end{array}\right).$$

Now, the last nonzero row has only one nonzero entry (to the left of the vertical bar), namely -9, so we don't need to do anything more for step 4. As for step 5, we scale R_3 by $-\frac{1}{9}$ to make the pivot equal to 1:

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & -5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -9 & -27 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & -3 & -5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

For step 6, we do $R_1 \to R_1 + 3R_3$ and $R_2 \to R_2 - R_3$:

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & -5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

For steps 7 and 8, we do $R_1 \to R_1 + 2R_2$:

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

Our matrix is now in reduced row echelon form. We can translate this matrix back to a very simple system of equations:

$$\begin{aligned} x &= 2 \\ y &= -1 \\ z &= 3\,. \end{aligned}$$