When is an Equation Separable?

A differential equation is **separable** if it can be written in the form

$$y' = \frac{dy}{dt} = \frac{g(t)}{f(y)},$$

where $f(y)$ and $g(t)$ are continuous functions of $y$ and $t$ respectively. We have written $y' = \frac{dy}{dt}$ since this form is helpful for understanding the method. Multiplying both sides by $f(y)$ and by $dt$ gives

$$f(y) \, dy = g(t) \, dt,$$

and we can then solve this via integration. Let’s do some examples to illustrate this.

**Examples for Separable Equations**

*Example 1.* Solve $\frac{dy}{dt} = t + 3 \frac{y^2}{y}$, where $y(0) = 2$.

We rearrange this equation such that the $y$’s are on one side and the $t$’s are on the other:

$$y^2 \, dy = (t + 3) \, dt,$$

and then we integrate both sides and solve for $y$:

$$\int y^2 \, dy = \int (t + 3) \, dt$$

$$\frac{1}{3} y^3 = \frac{1}{2} t^2 + 3t + C$$

$$y(t) = \sqrt[3]{\frac{3}{2} t^2 + 9t + D}.$$

Note that we technically multiply $C$ by 3, but rather than write it as $3C$, we can write it as another arbitrary constant. Some texts will even leave it as “$C$”. To solve for $D$, we plug in the initial condition $y(0) = 2$:

$$2 = y(0) = \sqrt[3]{\frac{3}{2} (0)^2 + 9(0) + D} = \sqrt[3]{D} \rightarrow D = 8,$$

so our solution is

$$y(t) = \sqrt[3]{\frac{3}{2} t^2 + 9t + 8}.$$

*Example 2.* Solve $\frac{dy}{dt} + ty = 0$, where $y(0) = 5$.

Let’s try and manipulate this equation to get $\frac{dy}{dt} = -\frac{g(t)}{f(y)}$. Subtracting $ty$ from both sides gives

$$\frac{dy}{dt} = -ty,$$

which is of the desired form, with $g(t) = t$ and $f(y) = \frac{1}{y}$. Now, we separate the terms, integrate, and solve for $y$:

$$\int \frac{dy}{y} = \int -t \, dt$$

$$\ln |y| = -\frac{1}{2} t^2 + C$$

$$y(t) = e^{-1/2t^2} + C.$$
First-Order Equations: Separable Equations

Note that via the exponent laws we can write

\[ y(t) = e^{-1/2t^2+C} = e^C e^{-1/2t^2} = De^{-1/2t^2}, \]

where \( D = e^C \) is an arbitrary constant (we used a similar rewriting of the constant in Example 1). Plugging in the initial condition \( y(0) = 5 \) gives \( D = 5 \), so our solution is

\[ y(t) = 5e^{-1/2t^2}. \]

Some Equations Aren’t Separable

Though the method of separable equations is a useful method, it rarely applies. Recall that we must be able to write the differential equation in the form

\[ \frac{dy}{dt} = \frac{g(t)}{f(y)} \]

in order to separate the variables \( t \) and \( y \). Sometimes this is not possible. For example, in the differential equation

\[ \frac{dy}{dt} + (\cos t)y = 1, \]

we cannot separate the \( y \)'s and \( t \)'s. This is because to isolate \( \frac{dy}{dt} \) we need to subtract \( (\cos t)y \) from both sides, but then \( 1 - (\cos t)y \) can’t be written in the form \( \frac{g(t)}{f(y)} \).

Despite this limitation, one can often determine relatively quickly whether a given differential equation is separable. It is usually worthwhile to check for this because if the equation is separable, then the method for solving it is straightforward compared to other methods.

DISCLAIMER: This handout uses notation and methods from the textbook commonly used for M 427J courses taught at the University of Austin:
Braun, Martin, *Differential Equations and Their Applications*, 4th ed. Springer
December 5, 1992.