First-Order Equations: Separable Equations

When is an Equation Separable?

A differential equation is **separable** if it can be written in the form

$$y' = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{g(t)}{f(y)}\,,$$

where $f(y)$ and $g(t)$ are continuous functions of y and t respectively. We have written $y' = \frac{\mathrm{d}y}{\mathrm{d}t}$ since this form is helpful for understanding the method. Multiplying both sides by $f(y)$ and by $\mathrm{d}t$ gives

$$f(y)\,\mathrm{d}y = g(t)\,\mathrm{d}t\,,$$

and we can then solve this via integration. Let's do some examples to illustrate this.

Examples for Separable Equations

Example 1. Solve $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t+3}{y^2}$, where $y(0) = 2$.

We rearrange this equation such that the y's are on one side and the t's are on the other:

$$y^2\,\mathrm{d}y = (t+3)\,\mathrm{d}t\,,$$

and then we integrate both sides and solve for y:

$$\begin{aligned}\int y^2\,\mathrm{d}y &= \int t + 3\,\mathrm{d}t\\ \frac{1}{3}y^3 &= \frac{1}{2}t^2 + 3t + C\\ y(t) &= \sqrt[3]{\frac{3}{2}t^2 + 9t + D}\,.\end{aligned}$$

Note that we technically multiply C by 3, but rather than write it as $3C$, we can write it as another arbitrary constant. Some texts will even leave it as "C". To solve for D, we plug in the initial condition $y(0) = 2$:

$$2 = y(0) = \sqrt[3]{\frac{3}{2}(0)^2 + 9(0) + D} = \sqrt[3]{D} \longrightarrow D = 8\,,$$

so our solution is

$$y(t) = \sqrt[3]{\frac{3}{2}t^2 + 9t + 8}.$$

Example 2. Solve $\frac{\mathrm{d}y}{\mathrm{d}t} + ty = 0$, where $y(0) = 5$.

Let's try and manipulate this equation to get $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{g(t)}{f(y)}$. Subtracting ty from both sides gives

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -ty\,,$$

which is of the desired form, with $g(t) = t$ and $f(y) = \frac{1}{y}$. Now, we separate the terms, integrate, and solve for y:

$$\begin{aligned}\frac{\mathrm{d}y}{y} &= -t\,\mathrm{d}t\\ \int \frac{\mathrm{d}y}{y} &= \int -t\,\mathrm{d}t\\ \ln|y| &= -\frac{1}{2}t^2 + C\\ y(t) &= e^{-1/2t^2+C}\,.\end{aligned}$$

Note that via the exponent laws we can write

$$y(t) = e^{-1/2t^2+C} = e^C e^{-1/2t^2} = De^{-1/2t^2}\,,$$

where $D = e^C$ is an arbitrary constant (we used a similar rewriting of the constant in Example 1). Plugging in the initial condition $y(0) = 5$ gives $D = 5$, so our solution is

$$y(t) = 5e^{-1/2t^2}\,.$$

Some Equations Aren't Separable

Though the method of separable equations is a useful method, it rarely applies. Recall that we must be able to write the differential equation in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{g(t)}{f(y)}$$

in order to separate the variables t and y. Sometimes this is not possible. For example, in the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} + (\cos t)y = 1\,,$$

we cannot separate the y's and t's. This is because to isolate $\frac{\mathrm{d}y}{\mathrm{d}t}$ we need to subtract $(\cos t)y$ from both sides, but then $1 - (\cos t)y$ can't be written in the form $\frac{g(t)}{f(y)}$.

Despite this limitation, one can often determine relatively quickly whether a given differential equation is separable. It is usually worthwhile to check for this because if the equation *is* separable, then the method for solving it is straightforward compared to other methods.

DISCLAIMER: This handout uses notation and methods from the textbook commonly used for M 427J courses taught at the University of Austin:
Braun, Martin, *Differential Equations and Their Applications*, 4th ed. Springer
December 5, 1992.