Second-Order Equations: Series Solutions

Summary

When dealing with a differential equation of the form

$$P(t)y'' + Q(t)y' + R(t)y = 0\,,$$

where $P(t), Q(t)$, and $R(t)$ are polynomials in t, first start by assuming

$$y(t) = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \cdots\,.$$

Then, compute y' and y'' and plug all three into your differential equation. The goal is to get the summations to have the same indices and powers of t. Once everything is in one summation sign, we can use the fact that all of the coefficients of t^n must equal 0 to get the **recurrence relation**. From there, we can get the first several terms of our series.

In initial value problems, $a_0 = y(0)$. This is because plugging in 0 to our solution $y(t)$, only the constant term, a_0, does not vanish. Similarly, $a_1 = y'(0)$. Try plugging 0 into the derivative $y'(t)$.

Examples

Example 1. Solve the initial-value problem $y'' + ty' + y = 0$ where $y(0) = 0$ and $y'(0) = 3$.

We have a homogeneous, linear, second-order equation whose coefficients are polynomials in t ($P(t) = R(t) = 1$ in this case). As previously stated, we assume that our solution has the form of a power series. Then, we differentiate to obtain expressions for y' and y'':

$$y = \sum_{n=0}^{\infty} a_n t^n$$

$$y' = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}\,.$$

Notice how the indices go up by 1 as we take derivatives, and that if we started the indices at 0, then the first term of y' would be 0 and the first two terms of y'' would be 0. Thus, increasing the index to 1 or to 2 respectively does not change the series. We now plug these expressions into the original equation:

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + t \sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = 0\,.$$

Bringing in the t to the second sum gives

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=1}^{\infty} n a_n t^{n} + \sum_{n=0}^{\infty} a_n t^n = 0\,.$$

Our next goal is to get the indices of the sums and the powers of t to match. We can decrease the index of the first sum from $n = 2$ to $n = 0$ if we replace every instance of n in the sum by $n + 2$. This gives

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^{n} + \sum_{n=1}^{\infty} n a_n t^{n} + \sum_{n=0}^{\infty} a_n t^n = 0\,.$$

Notice how this step gives us a t^n as well. We could have used the fact that $n = 0$ and $n = 1$ would make the first two terms of $n(n-1)a_n t^{n-2}$ equal to 0, but this would only change the index and not the exponent of t.

We *can* use this method for the second sum, however, since we don't want to change the exponent of t^n. Recall that changing the index of the first sum from $n = 1$ to $n = 0$ does not change the series itself, since $n = 0$ makes $na_nt^n = 0$. This gives

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=0}^{\infty} na_nt^n + \sum_{n=0}^{\infty} a_nt^n = 0\,.$$

Adding the last two sums together gives

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=0}^{\infty}(n+1)a_nt^n = 0\,.$$

We can combine these two sums as well, since they have the same powers of t and the same starting index:

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n+1)a_n\right]t^n = 0\,.$$

Now, on the left-hand side we have a polynomial (with possibly infinite degree). For this to be identically 0 as the equality requires, each of the coefficients must be 0. Thus, every term in the series must be 0, giving the equation

$$(n+2)(n+1)a_{n+2} + (n+1)a_n = 0, \text{ for all } n\,.$$

Such an equation (really a set of equations) is called a **recurrence relation**, since subsequent terms in the series rely implicitly on previous terms. To make this relation explicit, we solve this recurrence relation for a_{n+2}:

$$a_{n+2} = -\frac{a_n}{n+2}\,.$$

Suppose we are given a_0 and a_1. Then the next several coefficients are as follows:

$$a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{3}, a_4 = \frac{a_0}{8}, a_5 = \frac{a_1}{15}, a_6 = -\frac{a_0}{48}, a_7 = -\frac{a_1}{105}, \ldots\,.$$

Notice how we write a_4 in terms of a_0. This is because we already have a_2 in terms of a_0. We can then split our series into the terms with a_0 and the terms with a_1:

$$y(t) = a_0\left(1 - \frac{1}{2}t^2 + \frac{1}{8}t^4 - \frac{1}{48}t^6 + \cdots\right) + a_1\left(t - \frac{1}{3}t^3 + \frac{1}{15}t^5 - \frac{1}{105}t^7 + \cdots\right),$$

which gives two linearly independent solutions, one of which is the polynomial multiplied by a_0 and the other of which is the polynomial multiplied by a_1. Recall our initial conditions: $y(0) = 0$ and $y'(0) = 3$. When $t = 0$, $y(0) = a_0$, so $a_0 = 0$. Furthermore, $y'(0) = a_1$, so $a_1 = 3$. This gives us our particular solution:

$$y(t) = 3\left(t - \frac{1}{3}t^3 + \frac{1}{15}t^5 - \frac{1}{105}t^7 + \cdots\right).$$

Example 2. Solve the differential equation $(2-t)y'' + y' + (2+t)y = 0$ subject to $y(0) = 1$ and $y'(0) = 2$.

Again, we assume that y is a power series and plug in y, y', and y'', giving

$$(2-t)\sum_{n=2}^{\infty} n(n-1)a_nt^{n-2} + \sum_{n=1}^{\infty} na_nt^{n-1} + (2+t)\sum_{n=0}^{\infty} a_nt^n = 0\,.$$

Bringing in the $(2-t)$ to the first sum and $(2+t)$ to the third sum gives

$$\sum_{n=2}^{\infty} 2n(n-1)a_nt^{n-2} - \sum_{n=1}^{\infty} n(n-1)a_nt^{n-1} + \sum_{n=1}^{\infty} na_nt^{n-1} + \sum_{n=0}^{\infty} 2a_nt^n + \sum_{n=0}^{\infty} a_nt^{n+1} = 0\,.$$

Notice how for the second sum, $n = 1$ gives a term of 0, so we may write the index as $n = 1$ rather than $n = 2$. This allows us to combine the second and third sums, which yields

$$\sum_{n=2}^{\infty} 2n(n-1)a_n t^{n-2} - \sum_{n=1}^{\infty} n(n-2)a_n t^{n-1} + \sum_{n=0}^{\infty} 2a_n t^n + \sum_{n=0}^{\infty} a_n t^{n+1} = 0.$$

Decreasing the index of the first sum by 2 and the index of the second sum by 1 gives

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} t^n - \sum_{n=0}^{\infty} (n+1)(n-1)a_{n+1} t^n + \sum_{n=0}^{\infty} 2a_n t^n + \sum_{n=0}^{\infty} a_n t^{n+1} = 0\,.$$

To get a power of n for t in the last sum, we increase the index to $n = 1$, giving

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} t^n - \sum_{n=0}^{\infty} (n+1)(n-1)a_{n+1} t^n + \sum_{n=0}^{\infty} 2a_n t^n + \sum_{n=1}^{\infty} a_{n-1} t^n = 0\,.$$

We want all the series to have the same index and powers of t. To this end, we can simply write out the first term of each of the first three series to make the indices all equal to 1, giving

$$4a_2 + a_1 + 2a_0 + \sum_{n=1}^{\infty} 2(n+2)(n+1)a_{n+2} t^n - \sum_{n=1}^{\infty} (n+1)(n-1)a_{n+1} t^n + \sum_{n=1}^{\infty} 2a_n t^n + \sum_{n=1}^{\infty} a_{n-1} t^n = 0\,.$$

Putting all of the sums together gives

$$4a_2 + a_1 + 2a_0 + \sum_{n=1} \left[2(n+2)(n+1)a_{n+2} - (n+1)(n-1)a_{n+1} + 2a_n + a_{n-1}\right] t^n = 0\,.$$

Again, the left-hand side is a (possibly infinite degree) polynomial, all of whose coefficients must equal 0. Since the sum has no constant term, we get

$$4a_2 + a_1 + 2a_0 = 0 \longrightarrow a_2 = -\frac{a_1 + 2a_0}{4}\,.$$

The terms of the sum must equal 0, which gives

$$a_{n+2} = \frac{(n+1)(n-1)a_{n+1} - 2a_n - a_{n-1}}{2(n+2)(n+1)}\,.$$

Now, recall from the original example that our initial conditions for $y(0)$ and $y'(0)$ gave us a_0 and a_1 respectively. The same holds true here, so that $a_0 = 1$ and $a_1 = 2$. We compute the next few terms of the series:

$$\begin{aligned}
a_2 &= -\frac{a_1 + 2a_0}{4} = -1 \\
a_3 &= \frac{(2)(0)a_2 - 2a_1 - a_0}{2(3)(2)} = -\frac{5}{12} \\
a_4 &= \frac{(3)(1)a_3 - 2a_2 - a_1}{2(4)(3)} = \frac{-5/4 + 2 - 2}{24} = -\frac{5}{96} \\
&\vdots
\end{aligned}$$

We could go as far as we like, but there doesn't seem to be a distinguishable pattern. It is perfectly acceptable to write the final answer as

$$y(t) = 1 + 2t - t^2 - \frac{5}{12}t^3 - \frac{5}{96}t^4 + \cdots\,.$$

DISCLAIMER: This handout uses notation and methods from the textbook commonly used for M 427J courses taught at the University of Austin:
Braun, Martin, *Differential Equations and Their Applications*, 4$^{\text{th}}$ ed. Springer
December 5, 1992.