Maclaurin Series and Taylor Series

Summary

A **Taylor series** of a function $f(x)$ centered at $x = c$ gives a power series representation of $f(x)$. The formula for a Taylor series is given by

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(c)(x-c)^i}{i!},$$

where $f^{(i)}(c)$ denotes the i^{th} derivative of f evaluated at $x = c$ (or just $f(c)$ if $i = 0$). A **Maclaurin series** is a Taylor series with $c = 0$. The $\mathbf{n^{th}}$ **degree Taylor polynomial**, denoted $T_n(x)$, is the polynomial obtained by taking the first $n+1$ terms of this series (i.e. setting the upper bound equal to n).

What are Maclaurin and Taylor series?

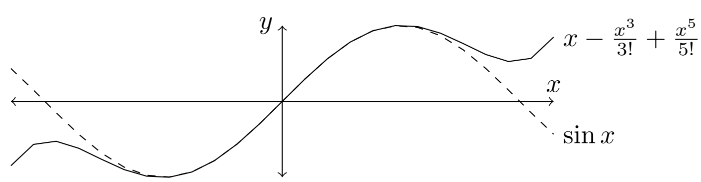
You may know that you can write many functions as power series, i.e. series of the form $\sum c_n x^n$, where c_n is a constant and x is a variable. For example, you can write the function $\frac{1}{1-x}$ in this form. In fact, you can represent any infinitely differentiable function by a power series. By infinitely differentiable, we mean a function $f(x)$ that has a first derivative, a second derivative, and so on. $(1+x)$, $\sin x$, and e^x are examples of such functions. Such representations are often infinite series, and thus are a bit unwiedly to use. For example, we will see that the Maclaurin series for $\sin x$ is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

If we were to take, say, the first 3 terms of this series, we would get an approximation of $\sin x$:

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

This is only an approximation: as $x \to \infty$, we see that this approximation goes off to $-\infty$, whereas $\sin x$ only varies between -1 and 1. In fact, this approximation is quite good for x values around the point $x = 0$. Below, we plot $\sin x$ (dashed) and the above approximation (solid):



We will first talk about how to get this Maclaurin series, then we will generalize to Taylor series.

Maclaurin series

A **Maclaurin series** for a function $f(x)$ is a power series representation of the function $f(x)$ around $x = 0$. We saw above that if we took only finitely many terms of the Maclaurin series, it was still a good approximation around $x = 0$. Later on, we will look at series that are good approximations around other values of x. For an infinitely differentiable function $f(x)$, its Maclaurin series is given by the formula

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)x^i}{i!},$$

where $f^{(i)}(0)$ denotes the i^{th} derivative of f at $x = 0$. We take $f^{(0)}(0)$ to just be $f(0)$, i.e. the original function value at $x = 0$. Writing a Maclaurin series for a function is simply a matter of computing the function's various derivatives and plugging them into the formula.

Example 1. Write a Maclaurin series for $f(x) = \sin x$.

First, we compute several derivatives of $\sin(x)$:

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \\ f^{(5)}(x) &= \cos x \\ &\vdots \end{aligned}$$

Note that we could keep going. In general, there's no fixed amount of times you should take the derivative. Next, we need to evaluate these various derivatives at $x = 0$:

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= 0 \\ f'''(0) &= -1 \\ f^{(4)}(0) &= 0 \\ f^{(5)}(0) &= 1 \\ &\vdots \end{aligned}$$

If we plug all these data into our formula, we get

$$\sin x = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)x^i}{i!} = \frac{(0)x^0}{0!} + \frac{(1)x^1}{1!} + \frac{(0)x^2}{2!} + \frac{(-1)x^3}{3!} + \frac{(0)x^4}{4!} + \frac{(1)x^5}{5!} + \cdots$$

Cleaning this up a bit, we get

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

You should compute as many derivatives as necessary so that it is clear for you how to get the last series representation. For example, it may be necessary for you to compute up to the 7^{th} derivative before you see the pattern in the terms, or you may only need to compute up to the 3^{rd} derivative.

Taylor series

A **Taylor series** is a more general form of the Maclaurin series in that it is still a power series representation of a function, but it may be "centered" at different x values. Recall that taking finitely many terms of the Maclaurin series gave a good approximation of the function around $x = 0$. For a Taylor series, we can choose this "center" so that taking finitely many terms of the Taylor series gives a good approximation around, say, $x = 2$ (here the center is $x = 2$). The Taylor series centered at $x = c$ for an infinitely differentiable function $f(x)$ is given by the formula

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(c)(x-c)^n}{i!}.$$

This formula differs from the previous one in two ways. Firstly, the function f and its derivatives are evaluated at $x = c$ rather than at $x = 0$. Secondly, rather than an x^i term we have an $(x - c)^i$ term. Looking at the formula, we see that a Maclaurin series is a Taylor series centered at $x = 0$ (i.e. with $c = 0$).

Example 2. Find a Taylor series representation centered at $x = 1$ for $f(x) = e^{-x}$.

As before, we compute the various derivatives of e^{-x}. We will compute three of them:

$$\begin{aligned} f(x) &= e^{-x} \\ f'(x) &= -e^{-x} \\ f''(x) &= e^{-x} \\ f'''(x) &= -e^{-x}. \end{aligned}$$

Next, we evaluate these derivatives at the center, $x = 1$:

$$\begin{aligned} f(1) &= e^{-1} \\ f'(1) &= -e^{-1} \\ f''(1) &= e^{-1} \\ f'''(1) &= -e^{-1} \end{aligned}$$

Finally, we plug this information into our formula:

$$f(x) = \frac{\left(e^{-1}\right)(x-1)^0}{0!} + \frac{\left(-e^{-1}\right)(x-1)^1}{1!} + \frac{\left(e^{-1}\right)(x-1)^2}{2!} + \frac{\left(-e^{-1}\right)(x-1)^3}{3!} + \cdots.$$

The series representation is then

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{e(n!)}.$$

When we take only the first $n+1$ terms of a Taylor series, we get what is called an **$\mathbf{n^{th}}$ degree Taylor polynomial**. In this case, we compute up to the n^{th} derivative of $f(x)$. This polynomial is sometimes denoted $T_n(x)$.

Example 3. Write $T_2(x)$ for the function $f(x) = \ln x$ centered at $x = 3$.

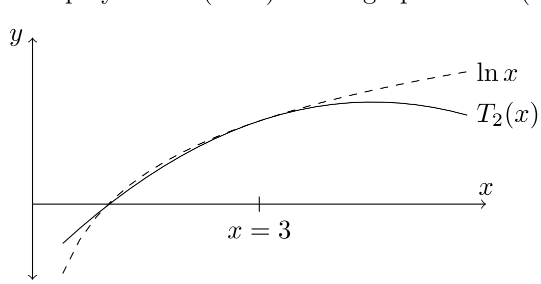
We compute the first and second derivatives of $f(x)$ and evaluate them at $x = 3$:

$$\begin{aligned} f(x) &= \ln x & f(3) &= \ln 3 \\ f'(x) &= \frac{1}{x} & f'(3) &= \frac{1}{3} \\ f''(x) &= -\frac{1}{x^2} & f''(3) &= -\frac{1}{9}. \end{aligned}$$

We then get

$$\begin{aligned} T_2(x) &= \frac{f(3)(x-3)^0}{0!} + \frac{f'(3)(x-3)^1}{1!} + \frac{f''(3)(x-3)^2}{2!} \\ &= \ln 3 + \frac{1}{3}(x-3) - \frac{1}{9}(x-3)^2. \end{aligned}$$

Let's compare the graph of this polynomial (solid) to the graph of $\ln x$ (dashed):



Notice how the polynomial $T_2(x)$ is a very good approximation around $x = 3$ but then fails to be a good approximation further away from $x = 3$.