Trigonometric Substitution

Summary

When integrating, Trig Sub is used to simplify expressions under radicals. We summarize which trigonometric substitutions to use in the following table:

Form	Relevant Identity	Substitution
$a^2 - b^2x^2$	$1 - \sin^2\theta = \cos^2\theta$	$x = \frac{a}{b}\sin\theta$
$a^2 + b^2x^2$	$1 + \tan^2\theta = \sec^2\theta$	$x = \frac{a}{b}\tan\theta$
$b^2x^2 - a^2$	$\sec^2\theta - 1 = \tan^2\theta$	$x = \frac{a}{b}\sec\theta$

Pythagorean Identities

The following Pythagorean identities will be important:

$$\cos^2 x + \sin^2 x = 1$$
$$\sec^2 x - \tan^2 x = 1\,,$$

and, for later use, we will write these as

$$1 - \sin^2 x = \cos^2 x \tag{1}$$
$$1 + \tan^2 x = \sec^2 x \tag{2}$$
$$\sec^2 x - 1 = \tan^2 x\,. \tag{3}$$

Using Trig Sub

Now let's consider the integral

$$\int_0^{1/2} \frac{1}{(1-x^2)^{\frac{3}{2}}}\,\mathrm{d}x\,.$$

It's not yet clear how to evaluate this integral. Trig Sub gives us a way to handle such integrals. Let's use the substitution

$$x = \sin\theta\,,$$
$$\mathrm{d}x = \cos\theta\,\mathrm{d}\theta\,.$$

We'll explain where this came from when we're done. Note how we compute $\mathrm{d}x$ similarly to how we compute $\mathrm{d}u$ in u-substitution. Also like with u-substitution, we now adjust our bounds: $x = 0$ goes to $\theta = \sin^{-1}(0) = 0$, and $x = \frac{1}{2}$ goes to $\theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Our integral then becomes

$$\begin{aligned}
\int_0^{1/2} \frac{1}{(1-x^2)^{\frac{3}{2}}}\,\mathrm{d}x &= \int_0^{\pi/6} \frac{1}{(1-(\sin\theta)^2)^{\frac{3}{2}}}(\cos\theta\,\mathrm{d}\theta) && \text{Substitution} \\
&= \int_0^{\pi/6} \frac{\cos\theta}{\left(1-\sin^2\theta\right)^{\frac{3}{2}}}\,\mathrm{d}\theta \\
&= \int_0^{\pi/6} \frac{\cos\theta}{(\cos^2\theta)^{\frac{3}{2}}}\,\mathrm{d}\theta && \text{Identity (1)} \\
&= \int_0^{\pi/6} \frac{\cos\theta}{\cos^3\theta}\,\mathrm{d}\theta \\
&= \int_0^{\pi/6} \sec^2\theta\,\mathrm{d}\theta && \sec\theta = \tfrac{1}{\cos\theta} \\
&= \tan\theta\Big|_0^{\pi/6} \\
&= \frac{\sqrt{3}}{3}\,.
\end{aligned}$$

The substitution allowed us to do the integral, but how did we know to use that expression for x? The key to the integral was using identity (1). Because the original integral had $1 - x^2$, replacing x with $\sin\theta$ ensures that we get (1), i.e. $1 - \sin^2\theta = \cos^2\theta$, which becomes much easier to work with when we have to take square roots as we did above.

Forms Beyond $\mathbf{1 - x^2}$

Let's now consider the integral

$$\int \sqrt{4 - x^2}\,\mathrm{d}x\,.$$

We omit the bounds from this example because we just want to focus on doing the substitution. The expression under the radical almost looks like $1 - x^2$, but there's a 4 instead of a 1. If we just plugged in $x = \sin\theta$, we would get $4 - \sin^2\theta$, which doesn't get us far. Let's instead use

$$\begin{aligned} x &= 2\sin\theta \\ \mathrm{d}x &= 2\cos\theta\,\mathrm{d}\theta\,. \end{aligned}$$

Then our substitution gives

$$\begin{aligned} \int \sqrt{4 - x^2}\,\mathrm{d}x &= \int \sqrt{4 - (2\sin\theta)^2}(2\cos\theta)\,\mathrm{d}\theta && \text{Substitution} \\ &= \int 2\cos\theta\sqrt{4 - 4\sin^2\theta}\,\mathrm{d}\theta\,. \end{aligned}$$

Factoring out a 4 from the radical gives

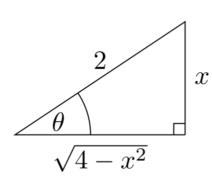
$$\begin{aligned} \int \sqrt{4 - x^2}\,\mathrm{d}x &= \int 2\cos\theta\sqrt{4\left(1 - \sin^2\theta\right)}\,\mathrm{d}\theta \\ &= \int 2\cos\theta\sqrt{4\cos^2\theta}\,\mathrm{d}\theta && \text{Identity (1)} \\ &= \int 4\cos^2\theta\,\mathrm{d}\theta\,. && \sqrt{4\cos^2\theta} = 2\cos\theta \end{aligned}$$

If we wanted to, we could then do this integral using $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$. After this step, and keeping in mind that $\sin(2\theta) = 2\sin\theta\cos\theta$, we end up with

$$\int \sqrt{4 - x^2}\,\mathrm{d}x = 2\theta + 2\sin\theta\cos\theta + C\,.$$

There's one problem though: we want this expression in terms of x, but it's still in terms of θ. If we look back at our substitution, we can get $\theta = \sin^{-1}\left(\frac{x}{2}\right)$ and $\sin\theta = \frac{x}{2}$. Figuring out what $\cos\theta$ is in terms of x requires a bit more work. This issue comes up frequently when doing Trig Sub.

Recall that, in a right triangle with angle θ, $\sin\theta$ is the ratio $\frac{O}{H}$, where O and H are the lengths of the opposite side and the hypotenuse respectively. Thus, the expression $\sin\theta = \frac{x}{2}$ gives us a right triangle with x for the opposite side length and 2 for the length of the hypotenuse. Using Pythagorean's Theorem allows us to determine that A, the length of the adjacent side, is $\sqrt{4 - x^2}$. Below we show the picture for this.



Using that $\cos\theta = \frac{A}{H}$ gives us $\cos\theta = \frac{\sqrt{4-x^2}}{2}$. Plugging all this in, we finish with

$$\int \sqrt{4 - x^2}\,\mathrm{d}x = 2\theta + 2\sin\theta\cos\theta + C = 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4 - x^2}}{4} + C\,.$$

We now look at a different kind of Trig Sub. Consider the integral

$$\int \frac{\sqrt{9x^2-16}}{x}\,\mathrm{d}x\,.$$

In the previous two examples, the expression under the radical was of the form $1-x^2$, whereas this expression is of the form x^2-1. If we think back to our Pythagorean identities (1)–(3), we find that (3) has the form x^2-1. The expression in this problem is not x^2-1 but $9x^2-16$, but if we choose a suitable coefficient in our substitution, we will end up with a coefficient times x^2-1 (think about how it didn't matter in the second example that it was $4-x^2$ rather than $1-x^2$). We use the substitution

$$\begin{aligned} x &= \frac{4}{3}\sec\theta\,,\\ \mathrm{d}x &= \frac{4}{3}\sec\theta\tan\theta\,\mathrm{d}\theta\,. \end{aligned}$$

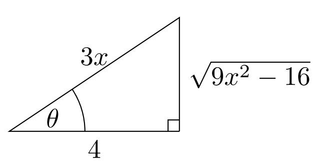
We use $\sec\theta$ rather than $\sin\theta$. If we plug this into our integral, we see that this was the correct substitution:

$$\begin{aligned} \int \frac{\sqrt{9x^2-16}}{x}\,\mathrm{d}x &= \int \frac{\sqrt{9\left(\frac{4}{3}\sec\theta\right)^2-16}}{\left(\frac{4}{3}\sec\theta\right)}\left(\frac{4}{3}\sec\theta\tan\theta\,\mathrm{d}\theta\right) && \text{Substitution}\\ &= \int \frac{\frac{4}{3}\sec\theta\tan\theta\sqrt{16\left(\sec^2\theta-1\right)}}{\frac{4}{3}\sec\theta}\,\mathrm{d}\theta\\ &= \int \tan\theta\sqrt{16\tan^2\theta}\,\mathrm{d}\theta && \text{Identity (3)}\\ &= \int 4\tan^2\theta\,\mathrm{d}\theta\,. \end{aligned}$$

Notice how the coefficient of $\frac{4}{3}$ we chose for $\sec\theta$ makes the substitution work. In general, for the expresion $b^2x^2-a^2$, we use the substitution $x=\frac{a}{b}\sec\theta$. Here, $a=\sqrt{16}=4$ and $b=\sqrt{9}=3$. To evaluate this integral we have now, we can appeal to another Pythagorean identity: $\tan^2\theta=\sec^2\theta-1$. Note that this is not one of the identities (1)–(3) since we won't use it for Trig Sub, but it is still useful for making certain integrals easier to evaluate. We now have

$$\int 4\tan^2\theta\,\mathrm{d}\theta = 4\int \sec^2\theta-1\,\mathrm{d}\theta = 4\tan\theta-4\theta+C\,.$$

As in the previous example, we wish for this to be in terms of x. We have the information that $x=\frac{4}{3}\sec\theta \rightarrow \sec\theta=\frac{3x}{4}$. Note that $\sec\theta=\frac{H}{A}$ since it is the reciprocal of $\cos\theta=\frac{A}{H}$. We now imagine a right triangle with angle θ, $H=3x$ and $A=4$. This gives us the following picture:



Reading off from this picture gives us that $\tan\theta=\frac{\sqrt{9x^2-16}}{4}$. Finally, from our $x=\frac{4}{3}\sec\theta$ we get $\theta=\sec^{-1}\left(\frac{3}{4}x\right)$. Plugging this all in, we get

$$\int \frac{\sqrt{9x^2-16}}{x}\,\mathrm{d}x = \sqrt{9x^2-16}-4\sec^{-1}\left(\frac{3}{4}x\right)+C\,.$$