

## What is a System of Linear Equations?

A **linear equation in two variables** is an equation of the form

$$ax + by = c,$$

where  $a$ ,  $b$ , and  $c$  are constants, and  $x$  and  $y$  are variables. For example,  $2x - 3y = 1$  and  $3x - 2y = 4$  are both linear equations in two variables. They are **linear** in the sense that each variable appears only to the first power (so  $x^2 + y = 1$  would not be linear). Linear equations can have any number of variables. The equation  $2x - y + z = 7$  would be a linear equation in three variables. The form above is called the **general form** of a linear equation.

A **system of equations** consists of two or more equations, and in a **system of linear equations**, these equations are all linear. For this handout, we'll consider systems of two linear equations with two variables.

We can solve any linear equation in two variables for  $y$ :

$$ax + by = c \longrightarrow y = -\frac{a}{b}x + \frac{c}{b},$$

which is the slope-intercept form of an equation for a line. Here,  $-\frac{a}{b}$  is the slope of our line, and  $\frac{c}{b}$  is our  $y$ -intercept. We can go from  $ax + by = c$  to slope-intercept form fairly quickly. Here's an example:

$$\begin{aligned} 2x - y &= 4. \\ -y &= -2x + 4 && \text{Subtract } 2x \text{ from both sides.} \\ y &= 2x - 4 && \text{Divide both sides by } -1. \end{aligned}$$

In this way, every linear equation in two variables corresponds to a line in the 2D plane, and a system of linear equations corresponds to a collection of lines, which are the lines for each equation in the system. After each problem we will graph the two equations on the same graph.

## Solving a System of Linear Equations

A **solution** to a system of equations is a pair  $(x, y)$  that satisfies both equations. For a point  $(x, y)$  to satisfy one of the equations means that it is a point on the line for that equation. So, if  $(x, y)$  solves  $2x - y = 4$  in our example above, then it lies on the line  $y = 2x - 4$ . If that point solves *both* equations in the system, then it lies on both of the lines. In other words, the solution to a system of linear equations is where the lines intersect. When considering whether two lines intersect, there are three possibilities:

1. The lines intersect at a single point.
2. The lines are parallel, and thus never intersect.
3. The lines are actually the same, and so intersect at infinitely many points.

In the following pages, we will work through an example for each of these three cases, using the algebraic methods of Substitution and Elimination.

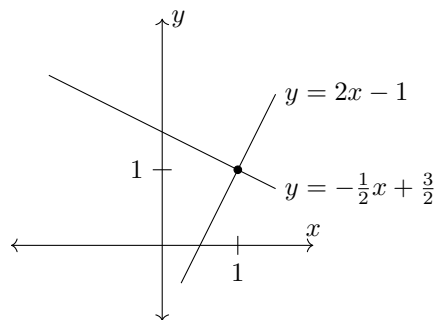
# Systems of Linear Equations with Two Variables

## Case 1: Systems with a Unique Solution

We'll work with the following system of equations:

$$\begin{aligned}2x - y &= 1 \\ 2y &= 3 - x\end{aligned}$$

Substitution	Elimination
<p>1. Choose one of the equations and solve for one of the variables. Here, we'll use the second equation. Solving for <math>x</math> gives</p> $x = -2y + 3.$	<p>1. First, arrange both equations so that the terms with the same variables line up:</p> $\begin{aligned}2x - y &= 1 \\ x + 2y &= 3.\end{aligned}$
<p>2. Put the value of this variable (in this case <math>x</math>) into the other equation. This gives</p> $\begin{aligned}2(-2y + 3) - y &= 1 \\ -5y + 6 &= 1.\end{aligned}$	<p>2. Multiply both sides of the first equation by 2, so that the <math>y</math>-coefficients are opposite between equations:</p> $\begin{aligned}4x - 2y &= 2 \\ x + 2y &= 3.\end{aligned}$
<p>3. Solve this equation for <math>y</math>:</p> $-5y = -5 \rightarrow y = 1.$	<p>3. Add the two equations together and solve for <math>x</math> to get</p> $5x = 5 \rightarrow x = 1.$
<p>4. Plug this value for <math>y</math> into either of the equations to get <math>x</math>. Here, we use the first equation:</p> $2x - 1 = 1 \rightarrow x = 1.$ <p>This gives the solution <math>(1, 1)</math>.</p>	<p>4. Plug this value for <math>x</math> into either of the equations to get <math>y</math>. Here, we use the second equation:</p> $2y = 2 \rightarrow y = 1.$ <p>This gives the solution <math>(1, 1)</math>.</p>



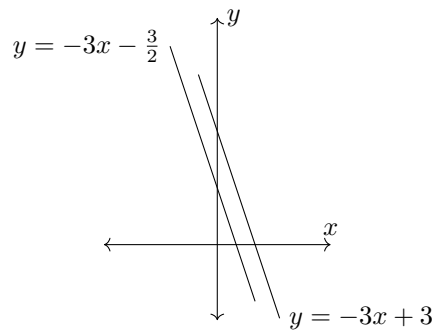
# Systems of Linear Equations with Two Variables

## Case 2: Systems with No Solution

We'll work with the following system of equations:

$$\begin{aligned}3x + y &= 3 \\ -6x - 2y &= 3.\end{aligned}$$

Substitution	Elimination
<p>1. Choose one of the equations and solve for one of the variables. Let's use the first equation to solve for <math>y</math>. This gives</p> $y = -3x + 3.$ <p>2. Plug in this value for <math>y</math> into the other equation. This gives</p> $\begin{aligned}-6x - 2(-3x + 3) &= 3 \\ -3 &= 3.\end{aligned}$ <p>3. We have a contradiction: we get that <math>-3 = 3</math>. When this happens, this indicates that the system has <i>no solution</i>.</p>	<p>1. Arrange both equations so that the terms with the same variables line up:</p> $\begin{aligned}3x + y &= 3 \\ -6x - 2y &= 3.\end{aligned}$ <p>2. Multiply both sides of the first equation by 2, so that the coefficients for the <math>x</math>-terms cancel out:</p> $\begin{aligned}6x + 2y &= 6 \\ -6x - 2y &= 3.\end{aligned}$ <p>3. If we try to add the two equations together, we get that <math>0 = 9</math>. This is a contradiction, which implies that the system has <i>no solution</i>, just as when we tried to solve using Substitution.</p>



# Systems of Linear Equations with Two Variables

## Case 3: Systems with Infinitely Many Solutions

We'll work with the following system of equations:

$$\begin{aligned}5x + 5y &= 10 \\ -x &= y - 2\end{aligned}$$

Substitution	Elimination
<p>1. Choose one of the equations and solve for one of the variables. We'll use the first equation to solve for <math>x</math>. This gives</p> $x = -y + 2.$ <p>2. Plug in this value for <math>x</math> into the other equation. This yields</p> $\begin{aligned}5(-y + 2) + 5y &= 10 \\ 10 &= 10.\end{aligned}$ <p>3. <math>10 = 10</math> is a true statement, but at first glance it doesn't seem to give us much information. In fact, getting an equation like this implies that the two equations in the system are logically equivalent: in other words, they are both satisfied by exactly the same <math>x</math>- and <math>y</math>-values. Thus, this system has infinitely many solutions.</p>	<p>1. Arrange both equations so that the same variables line up:</p> $\begin{aligned}5x + 5y &= 10 \\ -x - y &= -2.\end{aligned}$ <p>2. Multiply both sides of the second equation by 5, so that the coefficients for the <math>x</math>-terms will cancel:</p> $\begin{aligned}5x + 5y &= 10 \\ -5x - 5y &= -10.\end{aligned}$ <p>3. <math>0 = 0</math> is a true statement. As we saw in the Substitution method, such an equation implies that the two equations in the system are actually the same: notice how the second equation is just the first multiplied by <math>-5</math>. There are infinitely many solutions, namely all points on the line represented by the two equations.</p>

