Second-Order Equations: Variation of Parameters

The Method

The method of variation of parameters gives us a general method for solving nonhomogeneous linear second-order differential equations, namely those of the form

\[ y'' + p(t)y' + q(t)y = g(t). \]

Recall that to find a general solution for a nonhomogeneous equation takes two steps:

1. Find a general solution \( c_1 y_1(t) + c_2 y_2(t) \) for the homogeneous equation, \( y'' + p(t)y' + q(t)y = 0 \).

2. Find a particular solution \( \psi(t) \) to \( y'' + p(t)y' + q(t)y = g(t) \). Then the general solution is of the form \( c_1 y_1(t) + c_2 y_2(t) + \psi(t) \).

Variation of parameters allows us to find a particular solution \( \psi(t) \) given two linearly independent solutions \( y_1(t) \) and \( y_2(t) \) of the homogeneous equation. If we are given \( y_1(t) \) and \( y_2(t) \), then our particular solution, \( \psi(t) \), will be of the form

\[ \psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t). \]

We must figure out \( u_1(t) \) and \( u_2(t) \). To this end, we use two formulas (not proved here):

\[
\begin{align*}
    u'_1(t) &= -\frac{g(t)y_2(t)}{W[y_1, y_2](t)} \\
    u'_2(t) &= \frac{g(t)y_1(t)}{W[y_1, y_2](t)},
\end{align*}
\]

where \( W[y_1, y_2](t) \), also called the Wronskian, is given by

\[ W[y_1, y_2](t) = y_1(t)y'_2(t) - y'_1(t)y_2(t). \]

After computing \( u'_1 \) and \( u'_2 \), we can integrate both expressions to get \( u_1(t) \) and \( u_2(t) \) respectively. We illustrate this with an example.

**Example 1.** Write the general solution to \( y'' + y' - 2y = 4e^{2t} \).

The homogeneous equation is a second-order equation with constant coefficients. We can then compute the complementary solution

\[ y_c(t) = Ae^{-2t} + Be^t. \]

For the method, we only need two linearly independent solutions, so let’s use \( y_1(t) = e^{-2t} \) and \( y_2(t) = e^t \). With these, we compute the Wronskian:

\[ W[y_1, y_2](t) = (e^{-2t})(e^t) - (e^{-2t})(e^t) = e^{-t} + 2e^{-t} = 3e^{-t}. \]

With \( g(t) = 4e^{2t} \), we now compute

\[
\begin{align*}
    u'_1(t) &= -\frac{g(t)y_2(t)}{W[y_1, y_2](t)} \\
    &= -\frac{4e^{2t}}{3e^{-t}} \\
    &= -\frac{4}{3}e^{3t} \\
    u'_2(t) &= \frac{g(t)y_1(t)}{W[y_1, y_2](t)} \\
    &= \frac{4e^{2t}}{3e^{-t}} \\
    &= \frac{4}{3}e^t.
\end{align*}
\]
Next, we integrate both of these to get $u_1(t)$ and $u_2(t)$ respectively:

\[
\int u_1'(t) \, dt = -\frac{1}{3} \int 4e^{4t} \, dt \\
= -\frac{1}{3} e^{4t} + D_1 \\
\int u_2'(t) \, dt = \frac{4}{3} \int e^t \, dt \\
= \frac{4}{3} e^t + D_2.
\]

In fact, these constants of integration are unnecessary. We leave them in to see what happens to them—they will become irrelevant. We now compute

\[
\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \\
= \left(-\frac{1}{3} e^{4t} + D_1\right) (e^{-2t}) + \left(\frac{4}{3} e^t + D_2\right) (e^t) \\
= e^{2t} + D_1 e^{-2t} + D_2 e^t.
\]

Our general solution is then

\[
y(t) = (A + D_1)e^{-2t} + (B + D_2)e^t + e^{2t}.
\]

Since $A, B, D_1$, and $D_2$ are all arbitrary constants, we rewrite this as

\[
y(t) = c_1 e^{-2t} + c_2 e^t + e^{2t},
\]

which is why we could have ignored the constants of integration $D_1$ and $D_2$.

DISCLAIMER: This handout uses notation and methods from the textbook commonly used for M 427J courses taught at the University of Austin:

Braun, Martin, *Differential Equations and Their Applications*, 4th ed. Springer

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