Second-Order Equations: Variation of Parameters

The Method

The method of variation of parameters gives us a general method for solving nonhomogeneous linear second-order differential equations, namely those of the form

$$y'' + p(t)y' + q(t)y = g(t)\,.$$

Recall that to find a general solution for a nonhomogeneous equation takes two steps:

1. Find a general solution $c_1y_1(t) + c_2y_2(t)$ for the homogeneous equation, $y'' + p(t)y' + q(t)y = 0$.
2. Find a particular solution $\psi(t)$ to $y'' + p(t)y' + q(t)y = g(t)$. Then the general solution is of the form $c_1y_1(t) + c_2y_2(t) + \psi(t)$.

Variation of parameters allows us to find a particular solution $\psi(t)$ given two linearly independent solutions $y_1(t)$ and $y_2(t)$ of the homogeneous equation. If we are given $y_1(t)$ and $y_2(t)$, then our particular solution, $\psi(t)$, will be of the form

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t)\,.$$

We must figure out $u_1(t)$ and $u_2(t)$. To this end, we use two formulas (not proved here):

$$u_1'(t) = -\frac{g(t)y_2(t)}{W[y_1, y_2](t)}$$
$$u_2'(t) = \frac{g(t)y_1(t)}{W[y_1, y_2](t)}\,,$$

where $W[y_1, y_2](t)$, also called the **Wronskian**, is given by

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)\,.$$

After computing u_1' and u_2', we can integrate both expressions to get $u_1(t)$ and $u_2(t)$ respectively. We illustrate this with an example.

Example 1. Write the general solution to $y'' + y' - 2y = 4e^{2t}$.

The homogeneous equation is a second-order equation with constant coefficients. We can then compute the complementary solution

$$y_c(t) = Ae^{-2t} + Be^t.$$

For the method, we only need two linearly independent solutions, so let's use $y_1(t) = e^{-2t}$ and $y_2(t) = e^t$. With these, we compute the Wronskian:

$$W[y_1, y_2](t) = \left(e^{-2t}\right)\left(e^t\right) - \left(-2e^{-2t}\right)\left(e^t\right) = e^{-t} + 2e^{-t} = 3e^{-t}\,.$$

With $g(t) = 4e^{2t}$, we now compute

$$\begin{aligned} u_1'(t) &= -\frac{g(t)y_2(t)}{W[y_1, y_2](t)} \\ &= -\frac{\left(4e^{2t}\right)\left(e^t\right)}{\left(3e^{-t}\right)} \\ &= -\frac{4}{3}e^{4t} \\ u_2'(t) &= \frac{g(t)y_1(t)}{W[y_1, y_2](t)} \\ &= \frac{\left(4e^{2t}\right)\left(e^{-2t}\right)}{\left(3e^{-t}\right)} \\ &= \frac{4}{3}e^t\,. \end{aligned}$$

Next, we integrate both of these to get $u_1(t)$ and $u_2(t)$ respectively:

$$\begin{aligned}\int u_1'(t)\,\mathrm{d}t &= -\frac{1}{3}\int 4e^{4t}\,\mathrm{d}t \\ &= -\frac{1}{3}e^{4t} + D_1 \\ \int u_2'(t)\,\mathrm{d}t &= \frac{4}{3}\int e^{t}\,\mathrm{d}t \\ &= \frac{4}{3}e^{t} + D_2\,.\end{aligned}$$

In fact, these constants of integration are unnecessary. We leave them in to see what happens to them–they will become irrelevant. We now compute

$$\begin{aligned}\psi(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) \\ &= \left(-\frac{1}{3}e^{4t} + D_1\right)\left(e^{-2t}\right) + \left(\frac{4}{3}e^{t} + D_2\right)\left(e^{t}\right) \\ &= e^{2t} + D_1e^{-2t} + D_2e^{t}\,.\end{aligned}$$

Our general solution is then

$$y(t) = (A + D_1)e^{-2t} + (B + D_2)e^{t} + e^{2t}\,.$$

Since A, B, D_1, and D_2 are all arbitrary constants, we rewrite this as

$$y(t) = c_1e^{-2t} + c_2e^{t} + e^{2t},$$

which is why we could have ignored the constants of integration D_1 and D_2.

DISCLAIMER: This handout uses notation and methods from the textbook commonly used for M 427J courses taught at the University of Austin:
Braun, Martin, *Differential Equations and Their Applications*, 4th ed. Springer
December 5, 1992.