Introduction

We can compute basic integrals such as \( \int 2x^2 \, dx \) and \( \int (x^3 + 3) \, dx \) using sum and power rules and other basic formulas such as \( \int \frac{1}{x} \, dx = \ln x \). We cannot solve many integrals this way, however. For example, it is not clear how to use a sum or power rule to solve \( \int 2x \cos (x^2) \, dx \) or \( \int \frac{\ln x}{x} \, dx \). For such integrals, we must use the method of \( u \)-substitution.

The Method of \( u \)-Substitution

Let’s do the two examples given above, starting with \( \int 2x \cos (x^2) \, dx \). This is a good candidate for \( u \)-substitution because we have (1) a composition of functions (\( \cos x \) is composed with \( x^2 \)), and (2) the inner function’s derivative is outside (the derivative of \( x^2 \), \( 2x \), is on the outside of \( \cos (x^2) \)). We then set the inner function equal to \( u \) and compute \( du \) by deriving the right-hand side (with respect to \( x \)). Here, this gives

\[
u = x^2 \\
du = 2x \, dx.
\]

Remember the \( dx \) when deriving both sides. We can then substitute into our integral and proceed as normal, plugging \( x^2 \) back in for \( u \) at the end:

\[
\int 2x \cos (x^2) \, dx = \int \cos (x^2) (2x \, dx) \\
= \int \cos u \, du \\
= \sin u + C \\
= \sin (x^2) + C.
\]

Let’s do this for \( \int \frac{\ln x}{x} \, dx \). We can rewrite this as \( \int \frac{1}{x} \ln x \, dx \). Here, we don’t have a composition of functions, but we have a function, \( \ln x \), multiplied by its derivative, \( \frac{1}{x} \). This suggests using the substitution

\[
u = \ln x \\
du = \frac{1}{x} \, dx.
\]

As before, we can plug these data in straight away. For the future, we show another method for substitution. Rewrite the second line above as \( dx = x \, du \). This gives

\[
\int \frac{\ln x}{x} \, dx = \int \frac{u}{x} (x \, du) \\
= \int u \, du \\
= \frac{1}{2} u^2 + C \\
= \frac{1}{2} (\ln x)^2 + C.
\]

Working with bounds

So far, we have seen \( u \)-Substitution with indefinite integrals. Suppose now we have the definite integral \( \int_{0}^{\sqrt{\pi}} 2x \cos (x^2) \, dx \). We can proceed in one of two ways:

1. Change the bounds: First, we can change the bounds when doing our \( u \)-Substitution. Recall that we had \( u = x^2 \) for this example. Plugging the lower bound \( x = 0 \) into this gives a new lower bound of \( u = 0 \), and plugging the upper bound \( x = \sqrt{\pi} \) into this gives a new upper bound of \( u = \pi \). We then
evaluate the integral as normal:

\[
\int_0^{\sqrt{\pi}} 2x \cos (x^2) \, dx = \int_0^\pi \cos u \, du
\]

\[= \sin u \bigg|_0^\pi
\]

\[= \sin(\pi) - \sin(0)
\]

\[= 0.
\]

2. Don’t change the bounds: We can also just plug in our original bounds once we’ve rewritten the antiderivative in terms of \(x\). Recall that we found that

\[
\int 2x \cos (x^2) \, dx = \sin (x^2) + C,
\]

so evaluating this from \(x = 0\) to \(x = \sqrt{\pi}\) gives

\[
\int_0^{\sqrt{\pi}} 2x \cos (x^2) \, dx = \sin (x^2) \bigg|_0^{\sqrt{\pi}} = \sin(\pi) - \sin(0) = 0.
\]

We can use both methods, but sometimes changing the bounds is more convenient. Let’s consider our second example with bounds now: \(\int_1^{e^2} \frac{\ln x}{x} \, dx\). We can again proceed in two ways:

1. Change the bounds: Recall that in this integral we used the substitution \(u = \ln x\). Then \(x = 1\) goes to \(u = \ln(1) = 0\), and \(x = e^2\) goes to \(u = \ln (e^2) = 2\). In terms of \(u\), the antiderivative worked out to be \(\frac{1}{2} u^2\), so we plug in our bounds to get the answer:

\[
\int_1^{e^2} \frac{\ln x}{x} \, dx = \left. \frac{1}{2} u^2 \right|_0^2 = \frac{1}{2} (2^2 - 0^2) = 2.
\]

2. Don’t change the bounds: In terms of \(x\), our antiderivative was \(\frac{1}{2} (\ln x)^2\). Evaluating with our bounds gives

\[
\int_1^{e^2} \frac{\ln x}{x} \, dx = \left. \frac{1}{2} (\ln x)^2 \right|_1^{e^2} = \frac{1}{2} \left( (\ln (e^2))^2 - (\ln 1)^2 \right) = \frac{1}{2}(4) = 2.
\]