u-Substitution

Introduction

We can compute basic integrals such as $\int 2x^2\,\mathrm{d}x$ and $\int \left(x^3+3\right)\,\mathrm{d}x$ using sum and power rules and other basic formulas such as $\int \frac{1}{x}\,\mathrm{d}x = \ln x$. We cannot solve many integrals this way, however. For example, it is not clear how to use a sum or power rule to solve $\int 2x\cos\left(x^2\right)\,\mathrm{d}x$ or $\int \frac{\ln x}{x}\,\mathrm{d}x$. For such integrals, we must use the method of u-substitution.

The Method of u-Substitution

Let's do the two examples given above, starting with $\int 2x\cos\left(x^2\right)\,\mathrm{d}x$. This is a good candidate for u-substitution because we have (1) a composition of functions ($\cos x$ is composed with x^2), and (2) the inner function's derivative is outside (the derivative of x^2, $2x$, is on the outside of $\cos\left(x^2\right)$). We then set the inner function equal to u and compute $\mathrm{d}u$ by deriving the right-hand side (with respect to x). Here, this gives

$$\begin{aligned} u &= x^2 \\ \mathrm{d}u &= 2x\,\mathrm{d}x\,. \end{aligned}$$

Remember the $\mathrm{d}x$ when deriving both sides. We can then substitute into our integral and proceed as normal, plugging x^2 back in for u at the end:

$$\begin{aligned} \int 2x\cos\left(x^2\right)\,\mathrm{d}x &= \int \cos\left(x^2\right)(2x\,\mathrm{d}x) \\ &= \int \cos u\,\mathrm{d}u \\ &= \sin u + C \\ &= \sin\left(x^2\right) + C\,. \end{aligned}$$

Let's do this for $\int \frac{\ln x}{x}\,\mathrm{d}x$. We can rewrite this as $\int \frac{1}{x}\ln x\,\mathrm{d}x$. Here, we don't have a composition of functions, but we have a function, $\ln x$, multiplied by its derivative, $\frac{1}{x}$. This suggests using the substitution

$$\begin{aligned} u &= \ln x \\ \mathrm{d}u &= \frac{1}{x}\mathrm{d}x\,. \end{aligned}$$

As before, we can plug these data in straight away. For the future, we show another method for substitution. Rewrite the second line above as $\mathrm{d}x = x\,\mathrm{d}u$. This gives

$$\begin{aligned} \int \frac{\ln x}{x}\,\mathrm{d}x &= \int \frac{u}{x}(x\,\mathrm{d}u) \\ &= \int u\,\mathrm{d}u \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}(\ln x)^2 + C\,. \end{aligned}$$

Working with bounds

So far, we have seen u-Substitution with indefinite integrals. Suppose now we have the definite integral $\int_0^{\sqrt{\pi}} 2x\cos\left(x^2\right)\,\mathrm{d}x$. We can proceed in one of two ways:

1. *Change the bounds*: First, we can change the bounds when doing our u-Substitution. Recall that we had $u = x^2$ for this example. Plugging the lower bound $x = 0$ into this gives a new lower bound of $u = 0$, and plugging the upper bound $x = \sqrt{\pi}$ into this gives a new upper bound of $u = \pi$. We then

evaluate the integral as normal:

$$\begin{aligned}\int_0^{\sqrt{\pi}} 2x\cos\left(x^2\right)\,\mathrm{d}x &= \int_0^{\pi}\cos u\,\mathrm{d}u\\ &= \sin u\Big|_0^{\pi}\\ &= \sin(\pi)-\sin(0)\\ &= 0\,.\end{aligned}$$

2. *Don't change the bounds*: We can also just plug in our original bounds once we've rewritten the antiderivative in terms of x. Recall that we found that

$$\int 2x\cos\left(x^2\right)\,\mathrm{d}x = \sin\left(x^2\right)+C\,,$$

so evaluating this from $x=0$ to $x=\sqrt{\pi}$ gives

$$\int_0^{\sqrt{\pi}} 2x\cos\left(x^2\right)\,\mathrm{d}x = \sin\left(x^2\right)\Big|_0^{\sqrt{\pi}} = \sin(\pi)-\sin(0)=0\,.$$

We can use both methods, but sometimes changing the bounds is more convenient. Let's consider our second example with bounds now: $\int_1^{e^2}\frac{\ln x}{x}\,\mathrm{d}x$. We can again proceed in two ways:

1. *Change the bounds*: Recall that in this integral we used the substitution $u=\ln x$. Then $x=1$ goes to $u=\ln(1)=0$, and $x=e^2$ goes to $u=\ln\left(e^2\right)=2$. In terms of u, the antiderivative worked out to be $\frac{1}{2}u^2$, so we plug in our bounds to get the answer:

$$\int_1^{e^2}\frac{\ln x}{x}\,\mathrm{d}x = \frac{1}{2}u^2\Big|_0^2 = \frac{1}{2}\left(2^2-0^2\right)=2\,.$$

2. *Don't change the bounds*: In terms of x, our antiderivative was $\frac{1}{2}\left(\ln x\right)^2$. Evaluating with our bounds gives

$$\int_1^{e^2}\frac{\ln x}{x}\,\mathrm{d}x = \frac{1}{2}\left(\ln x\right)^2\Big|_1^{e^2} = \frac{1}{2}\left(\left(\ln\left(e^2\right)\right)^2-\left(\ln 1\right)^2\right)=\frac{1}{2}(4)=2\,.$$